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Neutron Diffusion - Spherical Harmonics Theory

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ABSTRACT

Approximate solutions of the Transport Equation are obtained for a large class of problems in neutron diffusion, using the Spherical Harmonics Method. This method gives rise to a sequence $\{\psi_{n-1}\}$ of approximations, the accuracy of which increase with n.

The polynomials closely associated with the Spherical Harmonics Method are investigated, in particular their asymptotic properties. The latter make it possible to obtain exact solutions by letting n approach infinity.

Matrix representations of the general solutions (of order n) are developed which greatly reduce the analytical and numerical work connected with particular solutions. At the same time the discussion of boundary and other physical conditions is simplified.

Some results obtained previously by the "integral theory" methods of Placzek, Frankel, Mark, and others, are derived by letting n approach infinity in the Spherical Harmonics approximations of order n. It is likely that "integral theory" results in general can be obtained by the methods of this report.

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THE SPHERICAL HARMONICS METHOD

1.1. The Transport Equation.

The study of the diffusion of neutrons in a physical medium may, under quite general assumptions regarding the properties of the medium, the scattering and other physical processes, be reduced to the investigation of an integro-differential equation, the so-called Transport Equation. Cf. G. Placzek and G. Volkoff, 1.* In this report we shall restrict the discussion to the particular diffusion model obtained when the following specific assumptions are made:

1) That the scattering of neutrons takes place without change in velocity so that each velocity can be treated independently of all others.

2) That the scattering, as well as the neutrons (of the velocity under consideration) emerging from sources which may be present, is isotropic.

Under these specific assumptions the Transport Equation may be written in the following form:

$$(1.1.1) \quad \psi'_t(\vec{r},\vec{\theta},t) + \mathrm{div}\left[\vec{\theta}\,\psi(\vec{r},\vec{\theta},t)\right] + \sigma\psi(\vec{r},\vec{\theta},t) = \frac{\sigma(1+f)}{4\pi}\int \psi(\vec{r},\vec{\theta},t)\,d\theta + \frac{\sigma}{4\pi}q(\vec{r}),$$

where $\psi(\vec{r},\vec{\theta},t)$, the "angular distribution", represents the number of neutrons per unit volume and per unit solid angle at $\vec{r}$, travelling in the direction $\vec{\theta}$, at time t, and where $q(\vec{r})$ represents the source strength per unit volume

* For references, see p. 86.

at $\vec{r}$. The integration in [redacted] over all directions $\vec{\vartheta}$.

In this theory the physical medium is characterized by the two parameters σ and $(1+f)$, σ being the inverse total mean free path of the neutrons, and $(1+f)$ the number of emerging neutrons per collision so that $f \geq -1$. In the process of eliminating the velocity v which does not appear in (1.1.1), time t acquired the dimensions of $1/\sigma$. Measuring lengths in units of cm, we then have σ in units of 1/cm and t in units of cm.

We shall restrict the investigation of (1.1.1) to time dependence of the type:

(1.1.2) $$\Psi(\vec{r},\vec{\vartheta},t) = \Psi(\vec{r},\vec{\vartheta})e^{\alpha t}, \qquad \alpha \text{ in } 1/\text{cm},$$

which includes besides the cases of exponential increase and decrease also the stationary case ($\alpha = 0$). Substituting (1.1.2) in (1.1.1) we have

(1.1.3) $$\mathrm{div}[\vec{\vartheta}\Psi(\vec{r},\vec{\vartheta})] + \bar{\sigma}\Psi(\vec{r},\vec{\vartheta}) = \frac{\bar{\sigma}(1+\bar{f})}{4\pi}\int\Psi(\vec{r},\vec{\vartheta})d\vartheta + \frac{\bar{\sigma}}{4\pi}\bar{q}(\vec{r}),$$

where $\bar{\sigma} = \sigma+\alpha$, $\bar{f} = \frac{\sigma f-\alpha}{\sigma+\alpha}$, and $\bar{q}(\vec{r}) = \frac{\sigma}{\sigma+\alpha}q(\vec{r})$. In the stationary case we have simply $\bar{\sigma} = \sigma$, $\bar{f} = f$, and $\bar{q}(\vec{r}) = q(\vec{r})$.

Although the SH-method, to be defined in the next paragraph, in principal can be applied to any geometry (C. Mark, 1), that is, directly to (1.1.3), we shall in this report specialize (1.1.3) to plane and spherical geometries, in which case the Transport Equation can

(1.1.4) $$\mu\psi'_r(r,\mu)+\sigma\psi(r,\mu) = \tfrac{1}{2}\sigma(1+f)\int_{-1}^{1}\psi(r,\mu)d\mu+\tfrac{1}{2}\sigma q(r),$$

(1.1.5) $$\mu\psi'_r(r,\mu)+\frac{1-\mu^2}{r}\psi'_\mu(r,\mu)+\sigma\psi(r,\mu) = \tfrac{1}{2}\sigma(1+f)\int_{-1}^{1}\psi(r,\mu)d\mu+\tfrac{1}{2}\sigma q(r)$$

where r is the distance in cm from the origin (a fixed plane and a fixed point respectively), $\mu=\cos\theta$, and θ the angle between the vector r and the direction of motion of a neutron at r. If we measure r in units of the total mean free path we may set $\sigma=1$ in (1.1.4) and (1.1.5).

1.2. Definition of the Spherical Harmonics Method.

If $\psi(r,\mu)$ is expandable in Spherical Harmonics in μ, that is, if $\psi(r,\mu)$ can be written in the form of a Legendre series

(1.2.1) $$\psi(r,\mu) = \tfrac{1}{2}\sum_{k=0}^{\infty}(2k+1)\psi_k(r)P_k(\mu),$$

where the coefficients $\psi_k(r)$, the "angular moments", are given by

(1.2.2) $$\psi_k(r) = \int_{-1}^{1}\psi(r,\mu)P_k(\mu)d\mu,$$

then the integro-differential equations (1.1.4) and (1.1.5) can be transformed into infinite systems of linear differential equations. We substitute (1.2.1) in (1.1.4) and (1.1.5), multiply left and right hand sides by $P_k(\mu)$, $k = 0,1,\ldots,$ and integrate with respect to μ from -1 to 1, obtaining respectively

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$$(1.2.3)\quad (k+1)D_r\psi_{k+1}(r)+kD_r\psi_{k-1}(r)+(2k+1)\psi_k(r)=\begin{cases}(1+\bar{f})\psi_0(r)+\bar{q}(r), & k=0,\\ 0, & k=1,2,\ldots,\end{cases}$$

$$(1.2.4)\quad (k+1)\left[D_r+\frac{k+2}{r}\right]\psi_{k+1}(r)+k\left[D_r-\frac{k-1}{r}\right]\psi_{k-1}(r)+(2k+1)\psi_k(r)=\begin{cases}(1+\bar{f})\psi_0(r)+\bar{q}(r), & k=0,\\ 0, & k=1,2,\ldots,\end{cases}$$

where r is in units of the mean free path.

We define the SH-approximations, or the ψ_{n-1}-approximations (the Spherical Harmonics approximations of order n-1) of $\psi(r,\mu)$, by 1) replacing (1.2.1) by

$$(1.2.5)\quad \psi(r,\mu) = \tfrac{1}{2}\sum_{k=0}^{n-1}(2k+1)\psi_k(r)P_k(\mu),$$

and by 2) requiring that

$$(1.2.6)\quad \psi_n(r) = \int_{-1}^{1}\psi(r,\mu)P_n(\mu)d\mu = 0.$$

This approximation procedure leads to the following finite systems of linear differential equations:

$$(1.2.7)\quad (k+1)D_r\psi^{(n)}_{k+1}(r)+kD_r\psi^{(n)}_{k-1}(r)+(2k+1)\psi^{(n)}_k(r)=\begin{cases}(1+\bar{f})\psi^{(n)}_0(r)+\bar{q}(r), & k=0,\\ 0, & k=1,2,\ldots(n-1),\end{cases}$$

$$(1.2.8)\quad (k+1)\left[D_r+\frac{k+2}{r}\right]\psi^{(n)}_{k+1}(r)+k\left[D_r-\frac{k-1}{r}\right]\psi^{(n)}_{k-1}(r)+(2k+1)\psi^{(n)}_k(r)=\begin{cases}(1+\bar{f})\psi^{(n)}_0(r)+\bar{q}(r), & k=0,\\ 0, & k=1,2,\ldots(n-1)\end{cases}$$

1.3. General Spherical Harmonics Solutions.

The solution of (1.2.7) subject to the condition (1.2.6), when $\bar{q}(r) = 0$, is given by

$$(1.3.1)\quad \psi^{(n)}_k(r) = \sum_{i=1}^{n/2}G_k(\bar{f},\mu_i)\left[(-1)^k A_i e^{r/\mu_i}+B_i e^{-r/\mu_i}\right],\ n \text{ even},$$

where $G_0(f,\mu) = 1$, $G_1(f,\mu) = -f\mu$, $G_n(f,\mu_i) = 0$, and

$$(1.3.2) \qquad kG_k(f,\mu) = (2k-1)\mu G_{k-1}(f,\mu) - (k-1)G_{k-2}(f,\mu),$$

and where A_i and B_i are unknowns to be determined by boundary and other physical conditions.

The solution of (1.2.8), when $\bar{q}(r) = 0$, is given by

$$(1.3.3) \qquad \psi_k^{(n)}(r) = \sum_{i=1}^{n/2} G_k(\bar{f},\mu_i)\left[(-1)^k A_i H_k(r/\mu_i) + B_i H_k(-r/\mu_i)\right], \quad n \text{ even},$$

where $H_0(x) = e^x/x$ and

$$(1.3.4) \qquad \left[D_r + \frac{k+2}{r}\right] H_{k+1}(ax) = \left[D_r - \frac{k-1}{r}\right] H_{k-1}(ax) = aH_k(ax).$$

The functions $H_k(x)$ are related to the Bessel functions of the second kind, denoted by $K_n(x)$. We have

$$(1.3.5) \qquad H_k(-x) = -\sqrt{\frac{2}{\pi x}}\, K_{k+\frac{1}{2}}(x).$$

See Janke and Emde 1, p.136.

The corresponding solutions, when $q(r) \neq 0$, are given by (n even)

$$(1.3.6) \qquad \psi_k^{(n)}(r) = \sum_{i=1}^{n/2} G_k(f,\mu_i)\Big\{(-1)^k\left[A_i + \tfrac{1}{2}\mu_i c_i \int^r q(r')e^{-r'/\mu_i}dr'\right]e^{r/\mu_i} + \left[B_i - \tfrac{1}{2}\mu_i c_i \int^r q(r')e^{r'/\mu_i}dr'\right]e^{-r/\mu_i}\Big\},$$

and

$$(1.3.7) \qquad \psi_k^{(n)}(r) = \sum_{i=1}^{n/2} G_k(f,\mu_i)\Big\{(-1)^k\left[A_i + \tfrac{1}{2}\mu_i c_i \int^r q(r')e^{-r'/\mu_i} \tfrac{r'}{\mu_i}dr'\right]e^{r/\mu_i} + \left[B_i - \tfrac{1}{2}\mu_i c_i \int^r q(r')e^{r'/\mu_i} r'/\mu_i\, dr'\right]e^{-r/\mu_i},$$

where the constant of integration has been absorbed in the unknown, and where $c_i = D_f(1/\mu_i^2)$. These solutions are given in C. Mark 2, pp. 7-15.

When n is odd $\psi_{n-1}(r)$ has an additional unknown. For the treament of this case, see Chapter IV, Note 1.

CHAPTER II

THE $G_n(f,\mu)$ - POLYNOMIALS

The general spherical harmonics solutions given in paragraph (1.3) all involved the polynomials $G_n(f,\mu)$. It has been shown by Mark that, irrespective of the geometry under consideration, these polynomials enter approximations of the spherical harmonics type. See C. Mark 1. In fact, the properties of these polynomials not only simplify the whole treatment but also make it possible to obtain the exact solution of the Transport Equation in some problems investigated using other techniques.

2.1. Definition of the $G_n(f,\mu)$ - Polynomials.

The following definitions of the polynomials $G_n(f,\mu)$ are equivalent, $-1 \leq \mu \leq 1$, $-1 \leq f < \infty$:

$$(2.1.1)\quad \begin{cases} G_o(f,\mu) = 1, \quad G_1(f,\mu) = -f\mu, \\ nG_n(f,\mu) = (2n-1)\mu G_{n-1}(f,\mu) - (n-1)G_{n-2}(f,\mu), \end{cases}$$

$$(2.1.2)\quad G_n(f,\mu) = P_n(\mu)-(1+f)\mu\left[\tfrac{1}{2}\log\left|\frac{1+\mu}{1-\mu}\right|P_n(\mu)-Q_n(\mu)\right],$$

where $P_n(\mu)$ and $Q_n(\mu)$ are Legendre functions of the first and second kind respectively, $P_o(\mu) = 1$, $P_1(\mu) = \mu$, $Q_o(\mu) = \frac{1}{2}\log\left|\frac{1+\mu}{1-\mu}\right|$, and $Q_1(\mu) = \mu Q_o(\mu) - 1$, and

$$(2.1.3)\quad G_n(f,\mu) = P_n(\mu)\left[1 - \tfrac{1}{2}(1+f)\mu\sum_i \frac{p_i}{\mu - \mu_i}\right],$$

where μ_i are the roots of $P_n(\mu)$ and p_i the corresponding weight numbers in Gauss' mechanical quadrature formula. In order to simplify our notation we drop the superscript n in $\mu_i^{(n)}$, and restrict ourselves to the case when n is even whenever the extention to the case when n is odd is immediate.

Since $G_n(f,\mu)$ has the same recurrence formula as $P_n(\mu)$ and $Q_n(\mu)$, two independent solutions of Legendre's equation, we have

(2.1.4) $$G_n(f,\mu) = aP_n(\mu) + bQ_n(\mu),$$

and hence

(2.1.5) $$G_o(f,\mu) = a + bQ_o(\mu) = 1,$$

$$G_1(f,\mu) = a\mu + b\, Q_o(\mu) - 1 = -f\mu.$$

Solving the two simultaneous equations (2.1.5) we obtain

(2.1.6) $$b = \mu(1+f), \text{ and } a = 1 - \mu(1+f)\tfrac{1}{2}\log\left|\frac{1+\mu}{1-\mu}\right|,$$

and hence

(2.1.7) $$G_n(f,\mu) = P_n(\mu) - (1+f)\mu\left[\tfrac{1}{2}\log\left|\frac{1+\mu}{1-\mu}\right|P_n(\mu) - Q_n(\mu)\right].$$

The equivalence of (2.1.2) and (2.1.3) can be shown using the integral representation of

(2.1.8) $$R_n(\mu) = \mu\left[\tfrac{1}{2}\log\left|\frac{1+u}{1-u}\right|P_n(\mu) - Q_n(\mu)\right],$$

given in G. Szegö 1, p. 76, i.e.,

(2.1.9) $$R_n(\mu) = \tfrac{1}{2}\mu\int_{-1}^{1}\frac{P_n(\mu) - P_n(t)}{\mu - t}\,dt.$$

Since the integrand is a polynomial of degree (n-1) in t, the integral is correctly given by Gauss' mechanical quadrature:

(2.1.10) $$R_n(\mu) = \tfrac{1}{2}\mu\sum\frac{P_n(\mu)-P_n(\mu_i)}{\mu - \mu_i}\,p_i = \tfrac{1}{2}\mu P_n(\mu)\sum_i \frac{p_i}{\mu - \mu_i}.$$

(2.1.11) $$G_n(f,\mu) = P_n(\mu)\left[1 - \tfrac{1}{2}(1+f)\mu\sum_i \frac{p_i}{\mu - \mu_i}\right].$$

2.2. Explicit formulas for $G_n(f,\mu)$, $n = 0,1,\ldots 8$.

$G_0 = 1$

$G_1 = -f\mu$

$G_2 = -1/2!\,(3f\mu^2+1)$

$G_3 = -1/3!\,\left[15f\mu^3+(5-4f)\mu\right]$

$G_4 = -1/4!\,\left[105f\mu^4+5(7-11f)\mu^2-9\right]$

$G_5 = -1/5!\,\left[945f\mu^5+105(3-7f)\mu^3-(161-64f)\mu\right]$

$G_6 = -9/6!\,\left[1155f\mu^6+35(11-34f)\mu^4-21(14-11f)\mu^2+25\right]$

$$G_7 = -9/7!\,\left[15015f\mu^7+385(13-50f)\mu^5-21(242-283f)\mu^3+ +(969-256f)\mu\right]$$

$$G_8 = -9/8!\,\left[225225f\mu^8+15015(5-23f)\mu^6-385(247-383f)\mu^4+ +3(9647-5053f)\mu^2-1225\right]$$

$G_0 = 1$

$G_1 = -\mu f$

$G_2 = -1/2!\,(3\mu^2f+1)$

$G_3 = -1/3!\,\left[(15\mu^3-4\mu)f+5\mu\right]$

$G_4 = -1/4!\,\left[(105\mu^4-55\mu^2)f+(35\mu^2-9)\right]$

$G_5 = -1/5!\,\left[(945\mu^5-735\mu^3+64\mu)f+(315\mu^3-161\mu)\right]$

$G_6 = -9/6!\,\left[(1155\mu^6-1190\mu^4+231\mu^2)f+(385\mu^4-294\mu^2+25)\right]$

$$G_7 = -9/7!\,\left[(15015\mu^7-19250\mu^5+5943\mu^3-256\mu)f+(5005\mu^5- -5082\mu^3+969\mu)\right]$$

$$G_8 = -9/8!\,\left[(225225\mu^8-345345\mu^6+147455\mu^4-15159\mu^2)f+ +(75075\mu^6-95095\mu^4+28941\mu^2-1225)\right]$$

2.3. The Elimination of f in $G_n(f,\mu_i)$, $R=$...

Theorem 2.3.1. The polynomials $G_n(f,\mu)$ can be written in the following form:

$$(2.3.1) \qquad G_n(f,\mu) = -1/n!\left[f\bar{S}_n(\mu) + \bar{R}_{n-2}(\mu)\right],$$

where $\bar{S}_n(\mu)$ and $\bar{R}_{n-2}(\mu)$ are polynomials of degree n and (n-2) respectively, $n = 0,1,\ldots,$ $\bar{R}_{-2}(\mu) = -1$, $\bar{S}_0(\mu) = 0$, $\bar{R}_{-1}(\mu) = 0$, and $\bar{S}_1(\mu) = \mu$.

Theorem 2.3.2. The recurrence formulas for $\bar{S}_n(\mu)$ and $\bar{R}_n(\mu)$ are

$$(2.3.2) \qquad \begin{aligned} \bar{S}_n(\mu) &= (2n-1)\mu\bar{S}_{n-1}(\mu) - (n-1)^2\bar{S}_{n-2}(\mu), \text{ and} \\ \bar{R}_n(\mu) &= (2n+3)\mu\bar{R}_{n-1}(\mu) - (n+1)^2\bar{R}_{n-2}(\mu). \end{aligned}$$

Theorem 2.3.3. The following "mixed" recurrence formula holds:

$$(2.3.3) \qquad \bar{S}_n(\mu)\bar{R}_{n-1}(\mu) - \bar{S}_{n+1}(\mu)\bar{R}_{n-2}(\mu) = (n!)^2\mu .$$

Above theorems are readily established using mathematical induction and the recurrence formula for $G_n(f,\mu)$.

Theorem 2.3.4. If μ_i are the non-negative roots of $G_n(f,\mu)$, $\mu_i > \mu_{i+1}$, $i = 1,2,\ldots[\frac{1}{2}(n+1)]$, then

$$(2.3.4) \qquad G_{n-1}(f,\mu_i) = c_{n-1}\mu_i/S_n(\mu_i),$$

where $c_n = 2^n(n!)^2/(2n+1)!$, and

$$(2.3.5) \qquad S_n(\mu) = 2^{n-1}[(n-1)!]\ \bar{S}_n(\mu)/(2n-1)!,$$

(so that the coefficient of the highest power of μ in $S_n(\mu)$ equals unity).

From $G_n(f,\mu_1) = 0$ follows

(2.3.6) $\quad f = -\bar{R}_{n-2}(\mu_1)/\bar{S}_n(\mu_1).$

Hence

(2.3.7) $\quad G_{n-1}(f,\mu_1) = 1/(n-1)!\left[\bar{S}_{n-1}(\mu_1)\frac{\bar{R}_{n-2}(\mu_1)}{\bar{S}_n(\mu_1)} - \bar{R}_{n-3}(\mu_1)\right],$

and applying Theorem 2.3.3 we obtain

(2.3.8) $\quad G_{n-1}(f,\mu_1) = 1/(n-1)!\ \frac{[(n-1)!]^2\mu_1}{\bar{S}_n(\mu_1)} = c_{n-1}\mu_1/S_n(\mu_1).$

Theorem 2.3.5. The parameter f in $G_k(f,\mu_1)$ can be eliminated, and we may write

(2.3.9) $\quad G_k(f,\mu_1) = c_k\mu_1/S_n(\mu_1)\cdot T^{(n)}_{k+1}(\mu_1),$

where $G_n(f,\mu_1) = 0$, $k = 0,1,\ldots n$, and $T^{(n)}_{k+1}(\mu)$ a polynomial of degree $n-k-1$.

This theorem is a consequence of the relations, $G_n(f,\mu_1) = 0$, $G_{n-1}(f,\mu_1) = c_{n-1}\mu_1/S_n(\mu_1)$, and

(2.3.10) $\quad (n-k+1)G_{n-k}(f,\mu) = (2n-2k+3)\mu\, G_{n-k+1}(f,\mu) -$
$\qquad\qquad -(n-k+2)G_{n-k+2}(f,\mu),$

an alternate form of (2.1.1), $k = 2,3,\ldots n$. We also have

(2.3.11) $\quad c_o = 1, \quad c_k = (k/2k+1)c_{k-1}, \quad k = 1,2,\ldots$

so that the coefficient of the highest power of μ in $T^{(n)}_{k+1}(\mu)$ equals unity.

We note that $\mu T^{(n)}_1(\mu) = S_n(\mu)$, and that $(1/3)T^{(n)}_2(\mu) = R_{n-2}(\mu) \equiv [2^{n-1}(n-1)!/(2n-1)!]\,\bar{R}_{n-2}(\mu).$

2.4. Explicit Formulas for $G_k(f,\mu_i^{(n)})$, $n = 2,3,\ldots 6$.

n = 2 $\quad S_2(\mu) = \mu^2,$

$$G_0(\mu_i) = c_0\mu_i/S_2(\mu_i)\cdot \mu_i \;,$$

$$G_1(\mu_i) = c_1\mu_i/S_2(\mu_i)\cdot 1 \;, \quad G_2(\mu_i) = 0, \; i = 1.$$

n = 3 $\quad S_3(\mu) = \mu^3 - \frac{4}{15}\mu \;,$

$$G_0(\mu_i) = c_0\mu_i/S_3(\mu_i)\cdot(\mu_i^2 - \frac{4}{15}),$$

$$G_1(\mu_i) = c_1\mu_i/S_3(\mu_i)\cdot \mu_i \;,$$

$$G_2(\mu_i) = c_2\mu_i/S_3(\mu_i)\cdot 1 \;, \quad G_3(\mu_i) = 0, \; i = 1,2.$$

n = 4 $\quad S_4(\mu) = \mu^4 - \frac{11}{21}\mu^2,$

$$G_0(\mu_i) = c_0\mu_i/S_4(\mu_i)\cdot(\mu_i^3 - \frac{11}{21}\mu_i),$$

$$G_1(\mu_i) = c_1\mu_i/S_4(\mu_i)\cdot(\mu_i^2 - \frac{9}{35}),$$

$$G_2(\mu_i) = c_2\mu_i/S_4(\mu_i)\cdot \mu_i \;,$$

$$G_3(\mu_i) = c_3\mu_i/S_4(\mu_i)\cdot 1 \;, \quad G_4(\mu_i) = 0, \; i = 1,2.$$

n = 5 $\quad S_5(\mu) = \mu^5 - \frac{7}{9}\mu^3 + \frac{64}{945}\mu \;,$

$$G_0(\mu_i) = c_0\mu_i/S_5(\mu_i)\cdot(\mu_i^4 - \frac{7}{9}\mu_i^2 + \frac{64}{945}),$$

$$G_1(\mu_i) = c_1\mu_i/S_5(\mu_i)\cdot(\mu_i^3 - \frac{23}{45}\mu_i),$$

$$G_2(\mu_i) = c_2\mu_i/S_5(\mu_i)\cdot(\mu_i^2 - \frac{16}{63}),$$

$$G_3(\mu_i) = c_3\mu_i/S_5(\mu_i)\cdot \mu_i \;,$$

$$G_4(\mu_i) = c_4\mu_i/S_5(\mu_i)\cdot 1 \;, \quad G_5(\mu_i) = 0, \; i = 1,2,3.$$

n = 6 $\quad S_6(\mu) = \mu^6 - \frac{34}{33}\mu^4 + \frac{1}{5}\mu^2 \;,$

$$G_0(\mu_i) = c_0\mu_i/S_6(\mu_i)\cdot(\mu_i^5 - \frac{34}{33}\mu_i^3 + \frac{1}{5}\mu_i),$$

$$G_1(\mu_i) = c_1\mu_i/S_6(\mu_i)\cdot(\mu_i^4 - \frac{42}{55}\mu_i^2 + \frac{5}{77}),$$

$$G_2(\mu_i) = c_2\mu_i/S_6(\mu_i)\cdot(\mu_i^3 - \frac{39}{77}\mu_i),$$

$\frac{n\,=\,6}{\text{cont.}}$

$$G_3(\mu_i) = c_3\mu_i/S_6(\mu_i)\cdot(\mu_i^2 - 25/99),$$

$$G_4(\mu_i) = c_4\mu_i/S_6(\mu_i)\cdot\ \mu_i\ ,$$

$$G_5(\mu_i) = c_5\mu_i/S_6(\mu_i)\cdot\ 1\ , \qquad G_6(\mu_i) = 0,\ i = 1,2,3.$$

2.5. General Formulas for $G_n(f,\mu)$.

The recurrence formula for $T_k^{(n)}(\mu)$, an immediate consequence of (1.2.1) and (2.3.9), is given by

$$(2.5.1) \qquad T_{k-2}^{(n)}(\mu) = \mu T_{k-1}^{(n)}(\mu) - g_{k-1}T_k^{(n)}(\mu),$$

where

$$(2.5.2) \qquad g_k = k^2/(2k-1)(2k+1).$$

Using the fact that $T_{n+1}^{(n)}(\mu) = 0$ and $T_n^{(n)}(\mu) = 1$, we then have

$$(2.5.3) \qquad \begin{aligned} T_{n-1}^{(n)}(u) &= \mu\ , \\ T_{n-2}^{(n)}(u) &= \mu^2 - g_{n-1}\ , \\ T_{n-3}^{(n)}(u) &= \mu^3 - \Big(\sum_{i=1}^{2} g_{n-i}\Big)\mu\ , \\ T_{n-4}^{(n)}(u) &= \mu^4 - \Big(\sum_{i=1}^{3} g_{n-i}\Big)\mu^2 + g_{n-1}g_{n-3}\ , \end{aligned}$$

and in general

$$(2.5.4) \qquad T_{n-k}^{(n)}(\mu) = \mu^k - \Big(\sum_{I_1} g_{n-i_1}\Big)\mu^{k-2} + \Big(\sum_{I_2} g_{n-i_1}g_{n-i_2}\Big)\mu^{k-4} - \cdots + \begin{cases} (-1)^{\frac{1}{2}k} \sum\limits_{I_{k/2}} g_{n-i_1}\cdots g_{n-i_{k/2}}, & k \text{ even} \\ (-1)^{\frac{1}{2}k-\frac{1}{2}}\mu \sum\limits_{I_{k/2-1/2}} g_{n-i_1}\cdots g_{n-i_{k/2}}, & k \text{ odd,} \end{cases}$$

where I_r , $r = 1,2,\ldots[\frac{1}{2}k]$, are r-tuples $(i_1\ ,\ i_2\ ,\ \ldots\ i_r)$ with $1 \leq i_1 \leq i_2-2 \leq i_3-4 \leq \ldots \leq i_r-2r+2 \leq k-2r+1$.

Formula (2.5.4) is readily obtained by induction using (2.5.1) and (2.5.3).

From a remark at the end of paragraph (2.3) we find that $G_n(f,\mu)$ can be written

(2.5.5) $G_n(f,\mu) = -[(2n)!/2^n(n!)^2]\cdot[\mu f T_1^{(n)}(\mu) + (1/3)T_2^{(n)}(\mu)]$.

Substituting $T_{n-k}^{(n)}(\mu)$ from (2.5.4) for $k = n-1$ and $k = n-2$ we obtain

(2.5.6) $$G_n(f,\mu) = -C_n\sum_{s=0}^{[n/2]}(-1)^s\mu^{n-2s}\Big[f\sum_{I_s}g_{n-i_1}\cdots g_{n-i_s} - g_1\sum_{J_{s-1}}g_{n-j}\cdots g_{n-j_{s-1}}\Big],$$

where $C_n = (2n)!/2^n(n!)^2$, and $\sum_{I_0} = \sum_{J_0} = 1$, $\sum_{J_1} = 0$, $\sum_{I_{n/2}} = 0$. The sets of integers I_r and J_r are the same as those defined in connection with formula (2.5.4), letting $k = n-1$ and $k = n-2$ respectively.

2.6. The Zeroes of $G_n(f,\mu)$.

From the preceding paragraphs, in particular from formula (2.5.6), it is clear that $G_n(f,u)$ is a polynomial of degree n in μ if $f \neq 0$, and of degree n-2 if $f = 0$. Furthermore, $G_n(f,\mu)$ is of the same parity as $P_n(\mu)$.

Case I, n even.

Theorem 2.6.1. If $(\mu_i^{(n)})^2$, $i = 1,2,\ldots\frac{1}{2}n$, are the squares of the roots of $G_n(f_1,\mu)$, $-1 \leq f_1 < \infty$, and if μ_i^2 are the corresponding roots of $G_n(-1,\mu) \equiv P_n(\mu)$, then

(2.6.1) $\mu_i^2 \leq (\mu_i^{(1)})^2 < \mu_{i-1}^2$, $i = 2,3,\ldots\frac{1}{2}n$,

(2.6.2) $\mu_1^2 \leq (\mu_1^{(1)})^2 < -1/f_1$, $-1 \leq f_1 < 0$,

(2.6.3) $-1/f_1 < (\mu_1^{(1)})^2 < 0$, $0 < f_1 < \infty$.

Theorem 2.6.2. If $-1 \leq f_2 < f_1 < \infty$, then

(2.6.4) $\mu_i^2 \leq (\mu_i^{(2)})^2 < (\mu_i^{(1)})^2 < \mu_{i-1}^2$, $i = 2,3,\ldots\frac{1}{2}n$,

(2.6.5) $\mu_1^2 \leq (\mu_1^{(2)})^2 < (\mu_1^{(1)})^2 < -1/f$, $-1 \leq f_2 < f_1 < 0$, In order to establish (2.6.1) we write (2.1.3) in the following form:

$$G_n(f,\mu)/P_n(\mu) = \left[1 - (1+f)\mu^2 \sum_{i=1}^{n/2} \frac{p_i}{\mu^2 - \mu_i^2}\right]. \tag{2.6.7}$$

If μ^2 varies from μ_{i+1}^2 to μ_i^2 , $i = 2,3,\ldots\frac{1}{2}n$, the function (2.6.7) increases from $-\infty$ to $+\infty$. From this we conclude that $G_n(f,\mu)$ changes sign at least once in each one of the intervals (μ_{i+1}^2, μ_i^2) for $f \neq -1$. This verifies (2.6.1) and accounts for n-2 roots of $G_n(f,\mu)$. The assumption that we have more than one root of $G_n(f,\mu)$ between successive roots of $P_n(\mu)$, e.g., three (the number must be odd) immediately leads to a contradiction of the fact that we can have at most n roots.

If μ^2 varies from μ_1^2 to ∞ , (2.6.7) increases from $-\infty$ to $\left[1 - (1+f)\sum_{i=1}^{n/2} p_i\right] = -f > 0$, if $-1 \leq f < 0$, which shows that for f in this range the remaining two zeroes of $G_n(f,\mu)$ are in the intervals (μ_1, ∞) and $(-\infty, -\mu_1)$. This argument and the following formula derived from (2.5.4):

$$\prod_{i=1}^{n/2} (\mu_i^{(1)})^2 = -\left(\prod_{i=1}^{n/2} g_{2i-1}\right)/f_1 , \quad n \text{ even}, \tag{2.6.8}$$

completes the proof of relation (2.6.2) since (2.6.8) implies

$$(\mu_1^{(1)})^2 < (\mu_1^{(1)})^2/\mu_1^2 \leq -1/f_1 , \quad -1 \leq f_1 < 0. \tag{2.6.9}$$

If f is in the interval $(0, \infty)$ we conclude that the remaining two zeroes are pure imaginary from formula (2.6.8). Further, if μ^2 varies from $-\infty$ to 0, the fraction (2.6.7) varies from $-f$ to 1, so that $G_n(f,\mu)$ regarded as a function of μ^2 changes sign in the interval $(-\infty, 0)$. This combined with (2.6.8) gives (2.6.3).

Theorem 2.6.2 can be shown to be true with an argument along these lines: If the derivative of the fraction (2.6.7) with respect to μ vanished for $\mu = \mu_0$, and if μ_0 substituted in (2.6.7) gave $f = f_0$, $-1 < f_0 < \infty$, we would conclude that the fraction possessed two zeroes in the μ_i-interval containing μ_0, for some f near f_0. This contradicts previous conclusions regarding the number of zeroes in each μ_i-interval.

<u>Case II, n odd.</u>

The following theorems, with the aid of

$$(2.6.10) \qquad \prod_{i=1}^{[n/2]} (\mu_i^{(1)})^2 = \prod_{i=1}^{[n/2]} g_{2i} - (g_1/f_1) \sum_{j_{\frac{n-3}{2}}} g_{n-j_1} \cdots g_{n-j_{\frac{n-3}{2}}}),$$

and the methods used in Case I, can readily be established:

Theorem 2.6.3. We have $\mu_{\frac{1}{2}n+\frac{1}{2}} = 0$, and

$$(2.6.11) \qquad \mu_i^2 \leq (\mu_i^{(1)})^2 < \mu_{i-1}^2 , \quad i = 2,3,\ldots \tfrac{1}{2}n-\tfrac{1}{2},$$

$$(2.6.12) \qquad \mu_1^2 \leq (\mu_1^{(1)})^2 < -1/f_1 , \quad -1 \leq f_1 < 0$$

$$(2.6.13) \qquad -1/f_1 < (\mu_1^{(1)})^2 \leq 0 , \quad 0 < f_1 \leq g_1 \frac{\sum_{j_{\frac{n-3}{2}}} g_{n-j_1} \cdots g_{n-j_{\frac{n-3}{2}}}}{\prod_{i=1}^{[n/2]} g_{2i}} .$$

Theorem 2.6.4. If $-1 \leq f_2 < f_1 < A$, then

$$(2.6.14) \qquad \mu_i^2 \leq (\mu_i^{(2)})^2 < (\mu_i^{(1)})^2 < \mu_{i-1}^2 , \quad i = 2,3,\ldots \tfrac{1}{2}n-\tfrac{1}{2} ,$$

$$(2.6.15) \qquad \mu_1^2 \leq (\mu_1^{(2)})^2 < (\mu_1^{(1)})^2 \quad -1/f_1 , \quad -1 \quad f_2 \quad f_1 \quad 0 ,$$

$$(2.6.16) \qquad -1/f_1 < (\mu_1^{(2)})^2 < (\mu_1^{(1)})^2 \leq 0 , \quad 0 < f_2 < f_1 \leq A ,$$

where A is the right hand side of (2.6.13).

2.7. Asymptotic Formulas for $G_n(f,\mu)$.

Theorem 2.7.1. If $\pm\mu$ are the zeroes of the transcendental equation

$$(2.7.1) \qquad (1+f) - \frac{1/\mu}{\operatorname{arcth}(1/\mu)} = 0, \quad -1 \leq f < \infty,$$

then $\pm\mu = \pm\lim_{n\to\infty} \mu_1^{(n)}$, where $\pm\mu_1^{(n)}$ are the "singular" zeroes of $G_n(f,\mu)$. The solutions of (2.7.1) are real if $-1 \leq f < 0$, and pure imaginary if $0 < f < \infty$. In the latter case we may replace μ by $|\mu|$ and arcth by arctan. In discussions where the shorter notation may be confusing, we denote μ by μ_0.

We observe that $\mu(-1) = 1$, $\mu(0) = \lim_{f\to 0^-} \mu(f) =$, $\lim_{f\to 0^+} |\mu|(f) = \infty$, and that $(1+f)|\mu| \sim 2/\pi$ if $f \to 0$. By expansion of (2.7.1) we have $\mu^2 \geq 1/-3f$, $-1 \leq f \leq 0$, and $|\mu|^2 \leq 1/3f$, $0 \leq f \leq \infty$. See (2.9.9). Consequently of the following derivatives the first is always positive and the second always negative:

$$(2.7.2) \qquad \mu'(f) = \frac{1}{\mu(1+f)} \, \frac{\mu^2 - 1}{1+f\mu^2}, \quad -1 < f < 0,$$

$$(2.7.3) \qquad |\mu|'(f) = \frac{-1}{|\mu|(1+f)} \, \frac{|\mu^2| + 1}{1-f|\mu^2|}, \quad 0 < f < \infty.$$

Theorem 2.7.2. For the "non-singular" zeroes of $G_n(f,\mu)$ we have the asymptotic forms:

$$(2.7.4) \qquad \mu_i = \cos\frac{2i-\frac{1}{2}}{2n+1}\pi + O(1/n^{\alpha}), \quad f = -1, \quad i = 1, 2, \ldots \left[\frac{n+1}{2}\right],$$

$$(2.7.5) \qquad \mu_i^{(1)} - \mu_i^{(2)} = \frac{\sqrt{1-\mu_i^2}}{n}\left[\arctan g(f_2,\mu_i) - \arctan g(f_1,\mu_i)\right] + O(1/n^{\alpha}),$$

where $\mu_i^{(1)}$ are the zeroes for $f = f_1$, $\mu_i^{(2)}$ those for $f = f_2$, $\alpha > 1$, and

$$(2.7.6) \qquad g(f,\mu) = \frac{1 - (1+f)\mu \operatorname{arcth} \mu}{(\pi/2)(1+f)\mu}, \quad 0 \le u \le 1.$$

As the derivative of $g(f,\mu)$ with respect to μ is always negative, $g(f,\mu)$ decreases monotonely from ∞ to $-\infty$, and $\arctan g(f,u)$ from $\pi/2$ to $-\pi/2$.

Theorem 2.7.3. We have

$$(2.7.7) \qquad \lim_{n\to\infty} \sum_{i=1}^{[\frac{n+1}{2}]} (\mu_i^{(1)} - \mu_i^{(2)}) F(\mu_i^{(1)}, \mu_i^{(2)}) = (\mu_0^{(1)} - \mu_0^{(2)}) F(\mu_0^{(1)}, \mu_0^{(2)}) + (1/\pi) \int_0^1 F(\mu) [\arctan g(f_2,\mu) - \arctan g(f_1,\mu)] d\mu$$

provided $F(\mu_i^{(1)}, \mu_i^{(2)})$ can be written as $F(\mu_i) + O(1/n^{\alpha}), \alpha > 0$, and the integral on the right hand side exists.

Theorem 2.7.4. We have

$$(2.7.8) \qquad \lim_{n\to\infty} \sum_{i=1}^{[\frac{n+1}{2}]} \mu_i^2 \frac{d}{df}(1/\mu_i^2) \cdot F(\mu_i) = \mu_0^2 \frac{d}{df}(1/\mu_0^2) \cdot F(\mu_0) - \int_0^1 \frac{d\mu}{[1-(1+f)\mu \operatorname{arcth} \mu]^2 + (\pi^2/4)(1+f)^2\mu^2}.$$

In particular, if $F(\mu_i) = G_k(f,\mu_i)$, we have from paragraph (3.1) - Note, that the right hand side of (2.7.8) equals $1/f$ for $k = 0$, and 0 for $k = 2, 4, \ldots$. This enables us to evaluate a number of definite integrals of the types (2.7.7) and (2.7.8).

Expansions.

$$(2.7.9) \quad \mu^2 = -(1/3f)(1 - \tfrac{4}{5}f + \tfrac{108}{175}f^2 - \tfrac{396}{875}f^3 + \tfrac{828}{2695}f^4 - \ldots),$$

$$(2.7.10) \quad 1/\mu^2 = -3f(1 + \tfrac{4}{5}f + \tfrac{4}{175}f^2 - \tfrac{4}{175}f^3 + \tfrac{7556}{336875}f^4 - \ldots),$$

$$(2.7.11) \quad \mu = \sqrt{(1/-3f)}(1 - \tfrac{2}{5}f + \tfrac{8}{35}f^2 - \tfrac{118}{875}f^3 + \tfrac{24778}{336875}f^4 - \ldots),$$

$$(2.7.12) \quad 1/\mu = \sqrt{-3f}\,(1 + \tfrac{2}{5}f - \tfrac{12}{175}f^2 + \tfrac{2}{125}f^3 + \tfrac{166}{67375}f^4 - \ldots),$$

$$(2.7.13) \quad f = -(1/3\mu^2)(1 + \tfrac{4}{15\mu^2} + \tfrac{44}{315\mu^4} + \tfrac{428}{4725\mu^6} + \tfrac{10636}{155925\mu^8} + \ldots$$

(2.7.14) $1/1+f = \frac{1}{2}\pi|\mu| - (|\mu|^2 - \frac{1}{3}|\mu|^4 + \frac{1}{5}|\mu|^6 - \ldots),$
$4/\pi - 1 < f < \infty,$

(2.7.15) $|\mu| = \frac{2}{\pi(1+f)} + \frac{2^3}{\pi^3(1+f)^2} + \frac{2\cdot 2^5}{\pi^5(1+f)^3} + \ldots,$
$0 < f < \infty,$

(2.7.16) $1/\mu \sim 1 - 2e^{-2/(1+f)}$, $1+f$ near 0.

We also have the following expansion for $\frac{d}{df}(1/\mu^2)$:

(2.7.16a) $\frac{d}{df}(1/\mu^2) = -3 + \frac{8}{5\mu^2} + \frac{212}{525\mu^4} + \frac{1584}{7875\mu^6} + \ldots$

<u>Proof of theorems.</u>

From Szegö 1,(8.21.3) and (8.71.19), we have for $\mu^2 > 1$:

(2.7.17) $P_n(\mu) \cong n^{-\frac{1}{2}}\left[\mu + \sqrt{\mu^2 - 1}\right]^n \varphi_1(\mu),$
(2.7.18) $Q_n(\mu) \cong n^{-\frac{1}{2}}\left[\mu + \sqrt{\mu^2 - 1}\right]^{-n-1} \varphi_2(\mu),$

where $\varphi_1(\mu)$, $\varphi_2(\mu)$, as well as their quotient, are bounded. Hence

(2.7.19) $G_n(\mu)/P_n(\mu) = \left[1-(1+f)\mu \operatorname{arcth} 1/\mu\right] + O\left[(\mu+\sqrt{\mu^2-1})\right]^{-2n-1}$

which shows that the convergence is uniform for $\mu^2 \geq 1+\varepsilon$. For a fixed f, $-1 < f < 0$, we choose $\varepsilon = \varepsilon(f)$, so that (2.7.19) is uniformly convergent in intervals containing the zeroes of $1-(1+f)\mu \operatorname{arcth} 1/\mu$.

Since (2.7.17) and (2.7.18) hold not only for $\mu^2 > 1$, but in the whole complex plane cut along the segment $[-1,1]$, similar arguments are valid for the case $\mu^2 \leq -\varepsilon$. Theorem (2.7.1) follows.

the following asymptotic expressions for $P_n(\mu)$ and $Q_n(\mu)$:

$$(2.7.20)\quad P_n(\cos\theta) = p(\theta)\cos(\overline{n+\tfrac{1}{2}}\theta-\pi/4) + O(1/n^{3/2}),$$

$$(2.7.21)\quad Q_n(\cos\theta) = -(\pi/2)p(\theta)\sin(\overline{n+\tfrac{1}{2}}\theta-\pi/4) + O(1/n^{3/2}),$$

where the bound for the error term holds uniformly in $\varepsilon \leq \theta \leq \pi-\varepsilon$, where $\cos\theta = \mu$, and $p(\theta) = 2\pi^{-\frac{1}{2}}(2n\sin\theta)^{-\frac{1}{2}}$. In the discussion below we restrict ourselves to μ in the interval $(0, 1-\delta)$, and θ in the interval $(\varepsilon, \pi/2)$, where $\cos\varepsilon = 1-\delta$.

Since $G_n(f,\mu)$ is a combination of $P_n(\mu)$ and $Q_n(\mu)$ with coefficients independant of n, we obtain the following asymptotic expression for $G_n(f,\mu)$:

$$(2.7.22)\quad G_n(f,\mu) = p(\theta)\left[(1-\overline{1+f}\mu\,\mathrm{arcth}\mu)\cos\theta-(\pi/2)(1+f)\mu\sin\theta\right] + O(1/n^{3/2}),$$

where the bound for the error term holds uniformly in $\varepsilon \leq \theta \leq \pi/2$.

The asymptotic positions of the zeroes of $G_n(f_i,\mu)$, $0 \leq \mu_i^{(i)} \leq 1-\delta$, $-1 < f < \infty$, are then given by

$$(2.7.23)\quad \tan(\overline{n+\tfrac{1}{2}}\theta_i^{(i)}-\pi/4) = \frac{1-(1+f_i)\mu_i^{(i)}\mathrm{arcth}\mu_i^{(i)}}{(\pi/2)(1+f_i)\mu_i^{(i)}} + O(1/n^{3/2}),$$

where $\cos\theta_i^{(i)} = \mu_i^{(i)}$.

Now, in view of formula (2.7.20), we may write

$$(2.7.24)\quad (\overline{n+\tfrac{1}{2}}\theta_i^{(i)}-\pi/4) = -(n+\tfrac{1}{2})(\theta_i-\theta_i^{(i)}) + (2i-1)\tfrac{\pi}{2} + O(1/n^{\frac{1}{2}}),$$

where $\cos\theta_i = \mu_i$, the roots of $P_n(\mu)$.

$$(2.7.25) \quad (\theta_1-\theta_1^{(1)}) = (1/n)\left[\pi/2 - \arctan\frac{1-\overline{1+f_1}\mu_1^{(1)}\operatorname{arcth}\mu_1^{(1)}}{\frac{1}{2}\pi(1+f_1)\mu_1^{(1)}}\right]+O(1/n^2),$$

so that $\theta_1-\theta_1^{(1)}$ and hence $\mu_1^{(1)}-\mu_1$ are of order 1/n. The second term in the bracket we denote by arctan $g(f_1,\mu_1)$ consistant with (2.7.6). The function arctan $g(f,\mu)$ is a well-behaved function in $0 \leq \mu \leq 1-\delta$, being bounded in this interval, having finite derivatives with respect to u of all orders, and possessing a Taylor expansion uniformly convergent (to this function) in $0 \leq u \leq 1-\delta$. We may then expand (2.7.6) about μ_1 and, noting that the remaining terms are of order $(1/n^2)$, keep only the first term, We obtain

$$(2.7.26) \quad (\theta_1-\theta_1^{(1)})) = (1/n)\left[\pi/2 - \arctan\frac{1-\overline{1+f_1}\mu_1\operatorname{arcth}\mu_1}{\frac{1}{2}\pi(1+f_1)\mu_1}\right]+O(1/n^2).$$

We now expand $\theta_1-\theta_1^{(1)}$ in terms of $\mu_1^{(1)}-\mu_1$ with the result that

$$(2.7.27) \quad (\theta_1-\theta_1^{(1)}) = (\mu_1^{(1)}-\mu_1)/\sin\theta_1 + O(1/n^2).$$

Substituting in (2.7.26) we have

$$(2.7.28) \quad (\mu_1^{(1)}-\mu_1) = \frac{\sqrt{1-\mu_1^2}}{n}\left[\pi/2 - \arctan g(f_1,\mu_1)\right] + O(1/n^2),$$

from which the statement (2.7.5), $0 \leq \mu_1^{(1)},\mu_1^{(2)} \leq 1-\delta$, follows immediately. The number of roots of $G_n(f,\mu)$ in the interval $(1-\delta,1)$ is of order n. From the inequalities in paragraph (2.6) we have that the sum of $(\mu_1^{(1)}-\mu_1)$ for μ_1's in the interval $(1-\delta,1)$ must be less than δ. Then, except for a finite number of such differences, $(\mu_1^{(1)}-\mu_1)$ must be of order δ/n, and (2.7.5) follows with the unessential restriction: "almost everywhere".

Formula (2.7.4) follows from G. Szegö 1, (6.3.8), where the inequality $\theta_i > \frac{2i-\frac{1}{2}}{2n+1}\pi$ is given, and from the fact that the sum of μ_i^2 as well as the sum of $\cos^2 \frac{2i-\frac{1}{2}}{2n+1}\pi$ equal $n/4 - 1/8 + O(1/n)$.

Theorem 2.7.3 follows from Theorem 2.7.2 and Szegö 1, (15.3.10); and Theorem 2.7.4 from (2.7.7) essentially by differentiation.

CHAPTER III

MATRIX REPRESENTATION OF THE ANGULAR MOMENTS

3.1. Plane Symmetry.

In plane symmetry the Transport Equation has the following form, when $q(r) = 0$. See (1.1.4).

$$\mu\Psi'_r(r,\mu)+\Psi(r,\mu) = \tfrac{1}{2}(1+f)\int_{-1}^{1}\Psi(r,\mu)d\mu, \tag{3.1.1}$$

where $\Psi(r,\mu)$ is the angular distribution, r the perpendicular distance from a fixed plane (the origin) measured in units of the mean free path, and μ the cosine of the angle between the direction of travel of a neutron at r and the vector r.

Applying the SH-method to (3.1.1) we obtain for the Ψ_{n-1}-approximation:

$$(k+1)D_r\Psi_{k+1}(r)+kD_r\Psi_{k-1}(r)+(2k+1)\Psi_k(r) = \begin{cases}(1+f)\Psi_0(r), & k=0,\\ 0, & k=1,2,\ldots\ldots(n-1).\end{cases} \tag{3.1.2}$$

The complete solution of (3.1.2) is given by (see Mark 1, p.)

$$\Psi_k(r) = \sum_{i=1}^{n/2} G_k(f,\mu)\left[(-1)^k A_i e^{r/\mu_i}+B_i e^{-r/\mu_i}\right], \quad k=0,1,\ldots(n-1), \tag{3.1.3}$$

which can readily be verified by substitution in (3.1.2), and by making use of (2.1.1). The quantities A_i and B_i are unknowns to be determined by boundary and other physical conditions. We refer to (3.1.3) as the exponential form or exp-form of the solution.

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We may also write (3.1.3) in an alternate form involving hyperbolic functions. We refer to this form as the hyperbolic or hyp-form of the solution, and have

$$(3.1.4)\quad \Psi_k(r) = (-1)^k \sum_{i=1}^{n/2} G_k(f,\mu_i) \Big\{ \left[A_i + (-1)^k B_i\right] \cosh r/\mu_i + \left[A_i + (-1)^{k+1} B_i\right] \sinh r/\mu_i \, .$$

We restrict the discussion in this chapter to the case when n is even. For the treatment of the odd case, see Chapter IV, Note 1.

Denoting vertical vectors by $[\;]_n$, and matrices by $(\;)_n$, or $(\;)_{n,m}$ if the horizontal dimension m is different from n, we can easily verify that the angular moments $\Psi_k(r)$ may be written in the vector form

$$(3.1.5)\quad \left[\Psi_k(r)\right]_n = C_n G_n K_n(\mu_i, r) \left[\bar{A}_i\right]_n ,$$

where

$$(3.1.6)\quad C_n = \begin{pmatrix} c_0 & 0 & \cdots & 0 \\ 0 & -c_1 & & \vdots \\ \vdots & & & 0 \\ 0 & \cdots & 0 & (-1)^{n-1} c_{n-1} \end{pmatrix}_n , \quad G_n = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & t_{23} \cdots & t_{2n} \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & t_{nn} \end{pmatrix}_n ,$$

with $c_0 = 1$, $c_k = (k/2k+1)c_{k-1}$, and t_{rs} the coefficients of μ^{n-s} in $T_r^{(n)}(\mu)$ [see (2.3.9) and paragraph (2.4)].

The form of $K_n(\mu_i, r)$ depends on the form in which the solution is given, as does the form of $[\bar{A}_i]$. Corresponding to the exp-form we have

$$K_n(\mu_i,r) = \begin{pmatrix} \mu_1^{n-1}\cdot e_1 \cdots\cdots & \mu_{n/2}^{n-1}\cdot e_{n/2} & \mu_1^{n-1}\cdot e_1 \cdots\cdots & \mu_{n/2}^{n-1}\cdot e_{n/2} \\ \mu_1^{n-2}\cdot e_1 \cdots\cdots & \mu_{n/2}^{n-2}\cdot e_{n/2} & -\mu_1^{n-2}\cdot e_1 \cdots\cdots & -\mu_{n/2}^{n-2}\cdot e_{n/2} \\ \mu_1^{n-3}\cdot e_1 \cdots\cdots & & \mu_1^{n-3}\cdot e_1 \cdots\cdots & \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ e_1 \cdots\cdots & e_{n/2} & -e_1 \cdots\cdots & -e_{n/2} \end{pmatrix}_n \tag{3.1.7}$$

with $e_i = e^{r/\mu_i}$ and $e_i = e^{-r/\mu_i}$; and corresponding to the hyp-form

$$K_n(\mu_i,r) = \begin{pmatrix} \mu_1^{n-1}\cdot ch_1 \cdots\cdots & \mu_{n/2}^{n-1}\cdot ch_{n/2} & \mu_1^{n-1}\cdot sh_1 \cdots\cdots & \mu_{n/2}^{n-1}\cdot sh_{n/2} \\ \mu_1^{n-2}\cdot sh_1 \cdots\cdots & \mu_{n/2}^{n-2}\cdot sh_{n/2} & \mu_1^{n-2}\cdot ch_1 \cdots\cdots & \mu_{n/2}^{n-2}\cdot ch_{n/2} \\ \mu_1^{n-3}\cdot ch_1 \cdots\cdots & & \mu_1^{n-3}\cdot sh_1 \cdots\cdots & \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ sh_1 \cdots\cdots & sh_{n/2} & ch_1 & ch_{n/2} \end{pmatrix}_n \tag{3.1.8}$$

with $ch_i = \cosh r/\mu_i$ and $sh_i = \sinh r/\mu_i$.

For the vector $[\bar{A}_i]_n$ we have respectively

$$\begin{aligned} \bar{A}_i &= (\mu_i/S_n(\mu_i))\cdot A_i\,, \\ \bar{A}_{i+n/2} &= (\mu_i/S_n(\mu_i))\cdot B_i\,, \end{aligned} \qquad i = 1,2,\ldots\tfrac{1}{2}n \tag{3.1.9}$$

and

$$\begin{aligned} \bar{A}_i &= (\mu_i/S_n(\mu_i))(A_i+B_i)\,, \\ \bar{A}_{i+n/2} &= (\mu_i/S_n(\mu_i))(A_i-B_i)\,, \end{aligned} \tag{3.1.10}$$

where $S_n(\mu_i)$ equals $\mu_i T_i^{(n)}(\mu_i)$.

We shall show that the following matrices are the inverses of (3.1.7) and (3.1.8):

$$K_n^{-1}(\mu_i,r) = \frac{1}{2}\begin{pmatrix} g_1 e_1 & \mu_1 g_1 e_1 & (\mu_1^2-s_1)g_1 e_1 & \mu_1(\mu_1^2-s_1)g_1 e_1 \cdots\cdots \\ g_2 e_2 & \mu_2 g_2 e_2 & (\mu_2^2-s_1)g_2 e_2 & \mu_2(\mu_2^2-s_1)g_2 e_2 \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ g_{n/2} e_{n/2} & \mu_{n/2} g_{n/2} e_{n/2} & (\mu_{n/2}^2-s_1)g_{n/2} e_{n/2} & \mu_{n/2}(\mu_{n/2}^2-s_1)g_{n/2} e_{n/2} \cdots\cdots \\ g_1 e_1 & -\mu_1 g_1 e_1 & (\mu_1^2-s_1)g_1 e_1 & -\mu_1(\mu_1^2-s_1)g_1 e_1 \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ g_{n/2} e_{n/2} & -\mu_{n/2} g_{n/2} e_{n/2} & (\mu_{n/2}^2-s_1)g_{n/2} e_{n/2} & -\mu_{n/2}(\mu_{n/2}-s_1)g_{n/2} e_{n/2} \cdots\cdots \end{pmatrix} \tag{3.1.11}$$

$$\begin{pmatrix} \cdots \left[\sum_{k=0}^{n/2-1}(-1)^k s_k \mu_1^{n-2k-2}\right] g_1 e_1 & \mu\left[\sum_{k=0}^{n/2-1}(-1)^k s_k \mu_1^{n-2k-2}\right] g_1 e_1 \\ \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \\ \cdots \left[\sum_{k=0}^{n/2-1}(-1)^k s_k \mu_{n/2}^{n-2k-2}\right] g_{n/2} e_{n/2} & \mu_{n/2}\left[\sum_{k=0}^{n/2-1}(-1)^k s_k \mu_{n/2}^{n-2k-2}\right] g_{n/2} e_{n/2} \\ \cdots \left[\sum(-1)^k s_k \mu_1^{n-2k-2}\right] g_1 e_1 & -\mu_1\left[\sum(-1)^k s_k \mu_1^{n-2k-2}\right] g_1 e_1 \\ \cdots\cdots & \cdots\cdots \\ \cdots \left[\sum(-1)^k s_k \mu_{n/2}^{n-2k-2}\right] g_{n/2} e_{n/2} & -\mu_{n/2}\left[\sum(-1)^k s_k \mu_{n/2}^{n-2k-2}\right] g_{n/2} e_{n/2} \end{pmatrix}_n$$

$$
K_n^{-1}(\mu_1, r) = \begin{pmatrix}
g_1 ch_1 & -\mu_1 g_1 sh_1 & (\mu_1^2 - s_1) g_1 ch_1 & -\mu_1(\mu_1^2 - s_1) g_1 sh_1 & \cdots \\
g_2 ch_2 & -\mu_2 g_2 sh_2 & (\mu_2^2 - s_1) g_2 ch_2 & -\mu_2(\mu_2^2 - s_1) g_2 sh_2 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
g_{n/2} ch_{n/2} & -\mu_{n/2} g_{n/2} sh_{n/2} & (\mu_{n/2}^2 - s_1) g_{n/2} ch_{n/2} & -\mu_{n/2}(\mu_{n/2}^2 - s_1) g_{n/2} sh_{n/2} & \cdots \\
-g_1 sh_1 & \mu_1 g_1 ch_1 & -(\mu_1^2 - s_1) g_1 sh_1 & \mu_1(\mu_1^2 - s_1) g_1 ch_1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-g_{n/2} sh_{n/2} & \mu_{n/2} g_{n/2} ch_{n/2} & -(\mu_{n/2}^2 - s_1) g_{n/2} sh_{n/2} & \mu_{n/2}(\mu_{n/2}^2 - s_1) g_{n/2} ch_{n/2} & \cdots
\end{pmatrix} \tag{3.1.12}
$$

$$
\begin{pmatrix}
\cdots & \left[\sum_{k=0}^{n/2-1}(-1)^k s_k \mu_1^{n-2k-2}\right] g_1 ch_1 & \mu_1\left[\sum_{k=0}^{n/2-1}(-1)^k s_k \mu_1^{n-2k-2}\right] g_1 sh_1 \\
\cdots & \cdots & \cdots \\
\cdots & \left[\sum_{k=0}^{n/2-1}(-1)^k s_k \mu_{n/2}^{n-2k-2}\right] g_{n/2} ch_{n/2} & \mu_{n/2}\left[\sum_{k=0}^{n/2-1}(-1)^k s_k \mu_{n/2}^{n-2k-2}\right] g_{n/2} sh_{n/2} \\
\cdots & \left[\sum(-1)^k s_k \mu_1^{n-2k-2}\right] g_1 sh_1 & \mu_1\left[\sum(-1)^k s_k \mu_1^{n-2k-2}\right] g_1 ch_1 \\
\cdots & \cdots & \cdots \\
\cdots & \left[\sum(-1)^k s_k \mu_{n/2}^{n-2k-2}\right] g_{n/2} sh_{n/2} & \mu_{n/2}\left[\sum(-1)^k s_k \mu_{n/2}^{n-2k-2}\right] g_{n/2} ch_{n/2}
\end{pmatrix}
$$

where $G_n(f,\mu_1) = a_n \sum_{k=0}^{n/2}(-1)^k s_k \mu_1^{n-2k}$, so that s_k is the symmetric function of the squares of the zeroes of $G_n(f,\mu)$, of order k; and where $g_1 = 2a_n/G_\mu'(f,\mu_1)$, which may also be written $g_1 = 1/\mu_1 \prod_{j\neq i}(\mu_1^2 - \mu_j^2)$, except in the case when $n = 2$ when we have $g_1 = 1/\mu_1$.

On multiplying the i^{th} row of K_n^{-1} into the i^{th} column of K_n taking the hyp-form of these matrices we obtain (modulus $\frac{1}{2}n$)

$$
g_1 \sum_{k=0}^{n/2-1}(-1)^k s_k (\tfrac{1}{2}n - k)\mu_1^{n-2k-1} = g_1 \frac{G_n'(f,\mu_1)}{2a_n} = 1 , \tag{3.1.13}
$$

and multiplying the i^{th} row into the j^{th} column, $j \neq i$, we have

$$
g_1(ch_1 ch_j - sh_1 sh_j)\mu_j/(\mu_j^2 - \mu_1^2)\sum_{k=0}^{n/2-1}(-1)^k s_k(\mu_j^{n-2k} - \mu_1^{n-2k}) = 0 \tag{3.1.14}
$$

for $i \leq \frac{1}{2}n$, $j \leq \frac{1}{2}n$, $j \neq i$; and similarly for the cases $i \geq \frac{1}{2}n$, $j \geq \frac{1}{2}n$; $i \leq \frac{1}{2}n$, $j \geq \frac{1}{2}n$; and $i \geq \frac{1}{2}n$, $j \leq \frac{1}{2}n$, $j \neq i$.

Note.

If we multiply K_n into K_n^{-1} the following formulas result, which we shall make use of in succeeding paragraphs:

$$(3.1.15)\quad \sum_{i=1}^{n/2} g_i \mu_i^{n-1} = 2a_n \sum_{i=1}^{n/2} \frac{\mu_i^{n-1}}{G_n'(f,\mu_i)} = 1,$$

$$(3.1.16)\quad 2a_n \sum_{i=1}^{n/2} \frac{\mu_i^{n-2k-1}}{G_n'(f,\mu_i)} = 0\,, \quad k = 1,2,\ldots(\tfrac{1}{2}n-1),$$

$$(3.1.17)\quad 2a_n \sum_{i=1}^{n/2} \frac{\mu_i^{n+2k-1}}{G_n'(f,\mu_i)} = \begin{cases} s_1, & k=1, \\ s_1^2 - s_2, & k=2, \\ s_1^3 - 2s_1 s_2 + s_3, & k=3, \\ s_1^4 - 3s_1^2 s_2 + 2s_1 s_3 + s_2^2 - s_4, & k=4, \\ \text{etc.} \end{cases}$$

Since $\mu_i'(f) = -\frac{d}{df} G_n(f,\mu_i)/G_n'(f,\mu_i)$ we have

$$(3.1.18)\quad \frac{d}{df}(1/\mu_i^2) = (2/\mu_i^3)\frac{d}{df}G_n(f,\mu_i)/G_n'(f,\mu_i) = \\ = -(2/\mu_i^3 n!)\bar{S}_n(\mu_i)/G_n'(f,\mu_i),$$

so that

$$(3.1.19)\quad \mu_i^2 \frac{d}{df}(1/\mu_i^2) G_k(f,\mu_i) = \\ = -\left[c_k(2n-1)!/2^{n-2} n!(n-1)!\right] \frac{T_{k+1}^{(n)}(\mu_i)}{G_n'(f,\mu_i)}.$$

Now $T_{k+1}^{(n)}(\mu)$ is a polynomial of degree n-k-1.

Hence

$$(3.1.20)\qquad \mu_i^2 \frac{d}{df}(1/\mu_i^2)\; G_k(f,\mu_i) = \begin{cases} 1/f, & k = 0, \\ 0, & k = 2,4,\ldots n, \end{cases}$$

and similarly

$$(3.1.21)\qquad \mu_i \frac{d}{df}(1/\mu_i^2)\; G_k(f,\mu_i) = \begin{cases} -1, & k = 1, \\ 0, & k = 3,5,\ldots(n-1) \end{cases}$$

where μ_i are the roots of $G_n(f,\mu)$.

In Chapter IV we shall need the matrix

$$K_n(\mu_1, r_2 - r_1) \equiv K_n(\mu_1, r_1) \cdot K_n^{-1}(\mu_1, r_2). \tag{3.1.22}$$

Introducing the functions $k_m \equiv k_m(\mu_1, r_2 - r_1)$ defined as follows:

$$k_m = \begin{cases} \sum_{i=1}^{n/2} g_i \mu_i^m \cosh(r_2 - r_1)/\mu_i , & m \text{ odd}, \\ -\sum_{i=1}^{n/2} g_i \mu_i^m \sinh(r_2 - r_1)/\mu_i , & m \text{ even}, \end{cases} \tag{3.1.23}$$

we have

$$K_n(\mu_1, r_2 - r_1) = \begin{pmatrix} k_{n-1} & k_n & k_{n+1} - s_1 k_{n-1} & k_{n+2} - s_1 k_n & \cdots & \sum_{K=0}^{n/2-1} (-1)^K s_K k_{2n-2K-2} \\ k_{n-2} & k_{n-1} & k_n - s_1 k_{n-2} & k_{n+1} - s_1 k_{n-1} & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ k_0 & k_1 & k_2 - s_1 k_0 & k_3 - s_1 k_1 & \cdots & \sum_{K=0}^{n/2-1} (-1)^K s_K k_{n-2K-1} \end{pmatrix}_n . \tag{3.1.24}$$

The computation of the elements in (3.1.24) is facilitated by making use of the formula

$$\sum_{K=0}^{n/2} (-1)^k s_k k_{n-2K} = 0. \tag{3.1.25}$$

They are in fact combinations of the elements in the first column. We have in the case n = 6:

$$K_6(\mu_1, r_2 - r_1) = \begin{pmatrix} k_5 & a_{34} + s_1 a_{21} & a_{24} & a_{36} - s_1 a_{21} & a_{26} & s_3 a_{21} \\ k_4 & a_{11} & a_{34} & a_{46} - s_2 a_{31} & a_{36} & s_3 a_{31} \\ k_3 & a_{21} & a_{11} - s_1 a_{31} & a_{56} - s_2 a_{41} & a_{46} & s_3 a_{41} \\ k_2 & a_{31} & a_{21} - s_1 a_{41} & a_{33} & a_{57} & s_3 a_{51} \\ k_1 & a_{41} & a_{31} - s_1 a_{51} & a_{43} & a_{33} + s_2 a_{51} & s_3 a_{61} \\ k_0 & a_{51} & a_{41} - s_1 a_{61} & a_{53} & a_{43} + s_2 a_{61} & a_{55} \end{pmatrix}_6 , \tag{3.1.26}$$

where the general element is denoted by a_{ij}.

3.2. Spherical Symmetry.

In spherical symmetry the Transport Equation has the following form, when $q(r) = 0$. See (1.1.5).

$$(3.2.1)\qquad \mu\Psi'_r(r,\mu)+\frac{1-\mu^2}{r}\Psi'_\mu(r,\mu)+\Psi(r,\mu) = \tfrac{1}{2}(1+f)\int_{-1}^{1}\Psi(r,u)du,$$

which after the SH-transformation becomes

$$(3.2.2)\qquad (k+1)\left[D_r+\frac{k+2}{r}\right]\Psi_{k+1}(r)+k\left[D_r-\frac{k-1}{r}\right]\Psi_{k-1}(r)+(2k+1)\Psi_k(r) = \begin{cases}(1+f)\Psi_0(r), & k = 0,\\ 0, & k = 1,2,\dots(n-1).\end{cases}$$

The solution of (3.2.2) is given by

(see Mark 1, p.)

$$(3.2.3)\qquad \Psi_k(r) = \sum_{i=1}^{n/2} G_k(f,\mu_i)\left[(-1)^k A_i H_k(r/\mu_i)+B_i H_k(-r/\mu_i)\right],\quad k=0,1,\dots(n-1),$$

where

$$(3.2.4)\qquad H_0(x) = (1/x)e^x,\quad H_1(x) = (1-1/x)(1/x)e^x,$$
$$H_k(x) = -[(2k-1)/x]H_{k-1}(x) + H_{k-2}(x),$$

and

$$(3.2.5)\qquad \left[D_r+\frac{k+2}{r}\right]H_{k+1}(ax) = \left[D_r-\frac{k-1}{r}\right]H_{k-1}(ax) = aH_k(ax),$$

$$(3.2.6)\qquad \begin{aligned} H_2(x) &= (1 - \frac{3}{x} + \frac{3}{x^2})H_0(x),\\ H_3(x) &= (1 - \frac{6}{x} + \frac{15}{x^2} + \frac{15}{x^3})H_0(x),\\ H_4(x) &= (1 - \frac{10}{x} + \frac{45}{x^2} - \frac{105}{x^3} + \frac{105}{x^4})H_0(x),\\ H_5(x) &= (1 - \frac{15}{x} + \frac{105}{x^2} - \frac{420}{x^3} + \frac{945}{x^4} - \frac{945}{x^5})H_0(x).\end{aligned}$$

Formula (3.2.6) can be derived by induction. Using (3.2.4), (3.2.5), and the recurrence formula for $G_n(f,\mu)$, the expression (3.2.3) can readily be shown to be a solution of (3.2.2).

The solution (3.2.3) may also be written in a form involving hyperbolic functions:

$$(3.2.7)\quad \Psi_k(r) = (-1)^k \sum_{i=1}^{2\ell} G_k(f,\mu_i)\Big\{ [A_i+(-1)^k B_i] H_k^{(1)}(r/\mu_i)\sinh r/\mu_i + [A_i+(-1)^{k+1} B_i] H_k^{(2)}(r/\mu_i)\cosh r/\mu_i \Big\},$$

where

$$(3.2.8)\quad \begin{aligned} H_0^{(1)}(x) &= H_0^{(2)}(x) = 1/x, \\ H_1^{(1)}(x) &= (1-\coth x/x)(1/x),\quad H_1^{(2)}(x) = (1-\tanh x/x)(1/x), \\ H_k^{(1)}(x) &= -[(2k-1)/x]\coth x\, H_{k-1}^{(2)}(x) + H_{k-2}^{(1)}(x), \\ H_k^{(2)}(x) &= -[(2k-1)/x]\tanh x\, H_{k-1}^{(1)}(x) + H_{k-2}^{(2)}(x). \end{aligned}$$

The angular moments in spherical geometry may also be written in vector form. We have

$$(3.2.9)\quad [\Psi_k(r)]_n = C_n T_n(r) Q_n(\mu_i,r)\,[\bar{A}_i]_n ,$$

where C_n is given by (3.1.6), $T_n(r)$ will be derived below, and $Q_n(\mu_i,r)$ is related to $K_n(\mu_i,r)$. We have in exp-form $Q_n(\mu_i,r) = (1/r)K_n(\mu_i,r)$, in hyp-form $Q_n(\mu_i,r) = (1/r)K_n(\mu_i,r)$ with ch_i and sh_i interchanged [see (3.1.8)], and $Q_n(\mu_i,r_2-r_1) = (r_2/r_1)K_n(\mu_i,r_2-r_1)$.

The vectors $[\bar{A}_i]_n$ in exp-form and hyp-form respectively are given by

$$(3.2.10)\quad \begin{aligned} \bar{A}_i &= (\mu_i^2/S_n(\mu_i))A_i , \\ \bar{A}_{i+\ell} &= -(\mu_i^2/S_n(\mu_i))B_i , \end{aligned}$$

and

$$(3.2.11\quad \begin{aligned} \bar{A}_i &= (\mu_i^2/S_n(\mu_i))(A_i+B_i) , \\ \bar{A}_{i+\ell} &= (\mu_i^2/S_n(\mu_i))(A_i-B_i) . \end{aligned}$$

Denoting the coefficients in the H_k-functions by h_{ij}, so that

$$(3.2.12) \qquad H_{i-1}(x) = \left(\sum_{j=1} h_{ij} x^{1-j}\right)(1/x)e^{-x},$$

and

$$(3.2.13) \qquad h_{ij} = -(2i-3)h_{i-1,j} + h_{i-2,j-2}\,, \quad h_{11} = 1,\; h_{21} = -1,\; h_{22} = 1,$$

we define the matrix $F_n(r)$ as follows:

$$(3.2.14) \qquad F_n(r) = \left(h_{ij} r^{j-1}\right) = \begin{pmatrix} 1 & 0 & 0 & \cdots & & \\ -1/r & 1 & 0 & \cdots & & \\ 3/r^2 & -3/r & 1 & 0 & \cdots & \\ -15/r^3 & 15/r^2 & -6/r & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ h_{n1} & \cdots & \cdots & \cdots & \cdots & h_{nn} \end{pmatrix}_n,$$

and have

$$(3.2.15) \qquad F_n^{-1}(r) = \left((-1)^{j-i} h_{j+1,2j-i+1}\, r^{j-i}\right) = \begin{pmatrix} 1 & 0 & 0 & \cdots & & & \\ 1/r & 1 & 0 & \cdots & & & \\ 0 & 3/r & 1 & 0 & \cdots & & \\ 0 & 3/r^2 & 6/r & 1 & 0 & \cdots & \\ 0 & 0 & 15/r^2 & 10/r & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \end{pmatrix}_n.$$

It is possible to construct matrices $E_n^{(i)}(r)$ so that

$$(3.2.16) \qquad E_n^{(i)}(r)F_n(r) = \begin{pmatrix} 0 & \cdots & 0 & \\ 0 & \cdots & 0 & \left(F_{n-2i-2}\right) \\ 0 & \cdots & 0 & \\ 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}_n.$$

We have $E_n^{(1)}(r) = I$, the unit matrix, and

$$(3.2.17) \qquad E_n^{(2)}(r) = \begin{pmatrix} 0 & 3/r & 1 & 0 & \cdots & & \\ 0 & 0 & 5/r & 1 & 0 & \cdots & \\ 0 & 0 & 0 & 7/r & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}_n,$$

$$E_n^{(3)}(r) = \begin{pmatrix} 0 & 0 & 15/r^2 & 10/r & 1 & 0 & \cdots & \\ 0 & 0 & 0 & 35/r^2 & 14/r & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}_n.$$

It is now possible to show that $T_n(r)$,

$$(3.2.18) \qquad T_n(r) = E_n(r)F_n(r),$$

where $E_n(r)$ is a combination of $E_n^{(i)}(r)$ with diagonal matrix coefficients which involve the coefficients of $T_{k+1}(\mu)$, $k=0,1,\ldots(n-1)$. See reference in connection with (3.1.6). In fact, $E_n(r)$ can be written

$$(3.2.19) \qquad E_n(r) = \sum_{i=1}^{n/2} T_n^{(i)}(r) E_n^{(i)}(r),$$

where $T_n^{(i)}(r)$ is a diagonal matrix with the elements $t_{1,2i-1}$, $t_{2,2i}$, $t_{3,2i+1}$, ... , $t_{n,2i+n-2}$, in the diagonal. The numbers t_{rs} are the coefficients of μ^{n-s} in $T_r^{(n)}(\mu)$ [see (2.3.9) and paragraph (2.4)].

Finally, we define $T_n(r_1,r_2)$ as follows:

$$(3.2.20) \qquad T_n(r_1,r_2) = T_n^{-1}(r_1)T_n(r_2).$$

3.3. Plane and Spherical Media with $\sigma = 0$.

In the plane case we obtain from (1.1.4) and (1.2.1) when $\sigma = 0$:

$$(3.3.1) \quad \psi_r'(r,\mu) = \frac{1}{2}\sum_{k=0}^{n-1}(2k+1)\psi_k'(r)P_k(\mu) = 0,$$

so that, after multiplying through by $P_k(\mu)$, $k = 0,1,\ldots(n-1)$, and integrating we have $\psi_k'(r) = 0$. As a consequence the angular moments and the angular distribution remain constant as r varies across a medium with $\sigma = 0$.

In the spherical case the equation (1.1.5) is reduced to

$$(3.3.2) \quad (k+2)\left[D_r+\frac{k+3}{r}\right]\psi_{k+2}(r) + (k+1)\left[D_r-\frac{k}{r}\right]\psi_k(r) = 0,$$

when $\sigma = 0$. The solution of (3.3.2) is readily found for $k = -1$ which yields $\psi_1(r)$ and for $k = (n-2)$ which gives $\psi_{n-2}(r)$, the other moments then follow by successive integrations. We have

$$(3.3.3) \quad \begin{aligned} \psi_0(r) &= \sum_{j=1,3..}^{n-1} A_j c_{1j} r^{j-1}, \quad c_{1j} = 1, \\ \psi_i(r) &= \sum_{j=i+1,i+3..}^{n-1} A_j c_{i+1,j} r^{j-1}, \quad i = 0,2,\ldots(n-2), \end{aligned}$$

and

$$(3.3.4) \quad c_{i+3,j} = -c_{i+1,j}\,\frac{i+1}{i+2}\,\frac{j-i-1}{j+i+2}\ .$$

Similarly

$$(3.3.5) \quad \begin{aligned} \psi_1(r) &= \sum_{j=2,4..}^{2} A_j c_{2j} r^{-j}, \quad c_{jj} = 1, \\ \psi_i(r) &= \sum_{j=2,4..}^{i+1} A_j c_{i+1,j} r^{-j}, \quad i = 1,3,\ldots(n-1), \end{aligned}$$

and

We now have the following matrix representation of $\Psi_k(r)$:

$$[\Psi_k(r)]_n = D_n(r)\,[A_i]_n, \tag{3.3.7}$$

where

$$D_n(r) \equiv \left(c_{ij}\, r^{(-1)^{i+j}\cdot(j-\frac{1}{2})-\frac{1}{2}}\right) = \tag{3.3.8}$$

$$= \begin{pmatrix} 1 & 0 & r^2 & 0 & r^4 & \cdots \\ 0 & 1/r^2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -1/5r^2 & 0 & -2/7r^4 & \cdots \\ 0 & 1/r^2 & 0 & 1/r^4 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1/21r^4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}_n,$$

and

$$D_n^{-1}(r) = \begin{pmatrix} 1 & 0 & 5 & 0 & 9 & \cdots \\ 0 & r^2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -5/r^2 & 0 & -30/r^2 & \cdots \\ 0 & -r^4 & 0 & r^4 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 21/r^4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}_n. \tag{3.3.9}$$

For applications we shall also need the matrices:

$$\begin{aligned} D_n^*(r_1, r_2) &= D_n(r_1)D_n^{-1}(r_2), \\ D_n(r_1, r_2) &= C_n^{-1}D_n^*(r_1, r_2)C_n, \end{aligned} \tag{3.3.10}$$

where C_n is defined by (3.1.6).

Denoting r_1^2/r_2^2 by s we have

$$D_n^*(r_1, r_2) = \begin{pmatrix} 1 & 0 & 5(1-s) & 0 & 9(1-\frac{10}{3}s+\frac{7}{3}s^2) & 0 & \cdots \\ 0 & 1/s & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & s & 0 & 6(1-s)s & 0 & \cdots \\ 0 & (1-1/s)/s & 0 & 1/s^2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & s^2 & 0 & \cdots \\ 0 & (1-\frac{14}{5s}+\frac{9}{5s^2})/s & 0 & \frac{14}{5}(1-1/s)/s^2 & 0 & 1/s^3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}_n, \tag{3.3.11}$$

and

$$D_n(r_1,r_2) =$$

$$= \begin{pmatrix} 1 & 0 & \frac{2}{3}(1-s) & 0 & \frac{8}{35}(1-\frac{10}{3}s+\frac{7}{3}s^2) & 0 & ,\cdots \\ 0 & 1/s & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & s & 0 & \frac{8}{7}(1-s)s & 0 & \cdots \\ 0 & \frac{36}{6s}(1-1/s)/s & 0 & 1/s^2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & s^2 & 0 & \cdots \\ 0 & \frac{231}{8s}(1-\frac{14}{5s}+\frac{9}{5s^2}) & 0 & \frac{693}{50s^2}(1-1/s) & 0 & 1/s^3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \qquad (3.3,12)$$

3.4. Plane and Spherical Media with f = 0.

When f = 0 we have $1/\mu_1 = 0$ for all n. We may obtain the solutions in this case from (3.1.4) and (3.2.7) by replacing $A_1 - B_1$ by $(A_1 - B_1)/\mu_1$ and let $1/\mu_1$ approach zero. Denoting $A_1 + B_1$ by A_0 and $(A_1 - B_1)/\mu_1$ by B_0, we have, making use of the fact that $G_k(f,\mu_1)$ is of order $1/\mu_1^k$:

$$(3.4.1) \quad \begin{aligned} \psi_0(r) &= A_0 + B_0 r + \sum_{i=2}^{n/2} G_0(f,\mu_1)\left[(A_1+B_1)\cosh r/\mu_1 + (A_1-B_1)\sinh r/\mu_1\right], \\ \psi_1(r) &= -B_0/3 - \sum_{i=2}^{n/2} G_1(f,\mu_1)\left[(A_1-B_1)\cosh r/\mu_1 + (A_1+B_1)\sinh r/\mu_1\right], \\ \psi_k(r) &= (-1)^k \sum_{i=2}^{n/2} G_k(f,\mu_1)\left\{\left[A_1+(-1)^k B_1\right]\cosh r/\mu_1 + \left[A_1+(-1)^{k+1}B_1\right]\sinh r/\mu_1\right\}, \\ & \qquad k = 2,3,\ldots(n-1), \end{aligned}$$

and

$$(3.4.2) \quad \begin{aligned} \psi_0(r) &= A_0 + B_0/r + \sum_{i=2}^{n/2} G_0(f,\mu_1)\left[(A_1+B_1)H_0^{(1)}(r/\mu_1)\sinh r/\mu_1 + (A_1-B_1)H_0^{(2)}(r/\mu_1)\cosh r/\mu_1\right], \\ \psi_1(r) &= B_0/3r^2 - \sum_{i=2}^{n/2} G_1(f,\mu_1)\left[(A_1-B_1)H_1^{(1)}(r/\mu_1)\sinh r/\mu_1 + (A_1+B_1)H_1^{(2)}(r/\mu_1)\cosh r/\mu_1\right], \\ \psi_k(r) &= B_0 k!/(2k+1)r^{k+1} + \\ &\quad + (-1)^k \sum_{i=2}^{n/2} G_k(f,\mu_1)\left\{\left[A_1+(-1)^k B_1\right]H_k^{(1)}(r,\mu_1)\sinh r/\mu_1 + \left[A_1+(-1)^{k+1}B_1\right]H_k^{(2)}(r,\mu_1)\cosh r/\mu_1\right\}. \end{aligned}$$

These solutions may also be written in exponential form corresponding to (3.1.3) and (3.2.3), and may of course also be obtained directly from the Transport Equation.

The matrix representations of the angular moments are again

$$(3.4.3)\qquad \begin{aligned} \left[\Psi_k(r)\right]n &= C_n G_n K_n(\mu_i,r)\left[\bar{A}_i\right]n\ , \text{ and} \\ \left[\Psi_k(r)\right]n &= C_n T_n(r) Q_n(\mu_i,r)\left[\bar{A}_i\right]n \end{aligned}$$

respectively. The matrices C_n, G_n, and $T_n(r)$ have the same form as before. The relation between K_n and Q_n is the same as before [see remarks in connection with (3.2.9)]. The K_n -matrices associated with the case when $f = 0$ have the following forms:

$$(3.4.4)\quad K_n(\mu_i,r) = \begin{pmatrix} 1 & r & \mu_2^{n-1}e_2 \cdots\cdots \mu_{n/2}^{n-1}e_{n/2} & \mu_2^{n-1}e_2 \cdots\cdots \mu_{n/2}^{n-1}e_{n/2} \\ 0 & 1 & \mu_2^{n-2}e_2 \cdots\cdots \mu_{n/2}^{n-2}e_{n/2} & -\mu_2^{n-2}e_2 \cdots\cdots -\mu_{n/2}^{n-2}e_{n/2} \\ 0 & 0 & \multicolumn{2}{c}{\multirow{4}{*}{$\begin{pmatrix} K_{n-2}(\mu_i,r) \\ i = 2,3,\ldots \tfrac{1}{2}n \end{pmatrix}$}} \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ 0 & 0 & & \end{pmatrix}_n,$$

$$(3.4.5)\quad K_n^{-1}(\mu_i,r) = \begin{pmatrix} 1 & -r & -s_1 & s_1 r \cdots\cdots\cdots -(-1)^{n/2}s_{n/2-1} & +(-1)^{n/2}s_{n/2-1}r \\ 0 & 1 & 0 & -s_1 \cdots\cdots\cdots 0 & -(-1)^{n/2}s_{n/2-1} \\ 0 & 0 & \multicolumn{3}{c}{\multirow{4}{*}{$\begin{pmatrix} K_{n-2}^{-1}(\mu_i,r) \\ i = 2,3,\ldots \tfrac{1}{2}n \end{pmatrix}$}} \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ 0 & 0 & & & \end{pmatrix}_n$$

with $G_n(0,\mu_i) \equiv a_{n-2}\sum_{k=0}^{n/2-1}(-1)^k s_k \mu_i^{n-2k-2}$, $s_o = 1$, and $g_i = 2a_{n-2}/G_n'(0,\mu_i)$, which may also be written $g_i = 1/\mu_i \prod_{j\neq i}(\mu_i^2-\mu_j^2)$, except in the case when $n = 4$ when we have $g_2 = 1/\mu_2$. In the hyp-form (3.4.5) remains unchanged but we have in place of (3.4.4):

$$(3.4.6)\quad K_n(\mu_i,r) = \begin{pmatrix} 1 & r & \mu_2^{n-1}ch_2 \cdots \mu_{n/2}^{n-1}ch_{n/2} & \mu_2^{n-1}sh_2 \cdots \mu_{n/2}^{n-1}sh_{n/2} \\ 0 & 1 & \mu_2^{n-2}ch_2 \cdots \mu_{n/2}^{n-2}ch_{n/2} & \mu_2^{n-2}sh_2 \cdots \mu_{n/2}^{n-2}sh_{n/2} \\ 0 & 0 & \multicolumn{2}{c}{\multirow{4}{*}{$\begin{pmatrix} K_{n-2}(\mu_i,r) \\ i = 2,3,\ldots \tfrac{1}{2}n \end{pmatrix}$}} \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ 0 & 0 & & \end{pmatrix}_n.$$

The vectors $[A_i]n$ are the same as those given by (3.1.9), (3.1.10), (3.2.10), and (3.2.11), except that in the plane case $\bar{A}_1 = A_0$, $\bar{A}_2 = B_0$, and in the spherical case $\bar{A}_1 = B_0$, $\bar{A}_2 = A_0$. Then

$$\text{(3.4.7)} \quad \begin{aligned} \bar{A}_{i+1} &= (\mu_i/S_n(\mu_i))(A_i+B_i), \\ \bar{A}_{i+\frac{1}{2}n} &= (\mu_i/S_n(\mu_i))(A_i-B_i), \end{aligned} \quad i = 2,3,\ldots\tfrac{1}{2}n,$$

and

$$\text{(3.4.8)} \quad \begin{aligned} \bar{\bar{A}}_{i+1} &= (\mu_i^2/S_n(\mu_i))(A_i+B_i), \\ \bar{\bar{A}}_{i+\frac{1}{2}n} &= (\mu_i^2/S_n(\mu_i))(A_i-B_i), \end{aligned}$$

in plane and spherical symmetry, when the solutions are given in hyp-form. By analogy we obtain the vector elements corresponding to (3.1.9) and (3.2.10), that is, the vector elements that go with the exp-form of the solutions.

Finally,

$$K_n(\mu_i, r_2-r_1) =$$

$$\text{(3.4.9)} \quad \left(\begin{array}{cc|cccc} 1 & -(r_2-r_1) & -s_1+k_{n-1} & s_1(r_2-r_1)+k_n & s_2+k_{n+1}+s_1k_{n-1} & \cdots\cdots \\ 0 & 1 & k_{n-2} & -s_1+k_{n-1} & k_n+s_1k_{n-2} & \cdots\cdots \\ 0 & 0 & & & & \\ \cdot & \cdot & & K_{n-2}(\mu_i, r_2-r_1) & & \\ \cdot & \cdot & & i = 2,3,\ldots\tfrac{1}{2}n & & \\ 0 & 0 & & & & \end{array}\right)_n$$

where k_m is defined by (3.1.23) except that the summation extends from 2 to $\frac{1}{2}n$ rather than 1 to $\frac{1}{2}n$.

CHAPTER IV

APPLICATIONS OF MATRIX REPRESENTATION OF $\psi_k(r)$ TO PROBLEMS IN NEUTRON DIFFUSION

We shall limit ourselves in this chapter to the discussion of three-medium problems. These give not only the formulas for the two-medium problems as a special case, but enable us by analogy to arrive at the formulas and techniques which go with the N-medium problems. We repeat here, that as far as our theory is concerned, plane and spherical media are characterized by the two parameters f and σ. In the plane case, however, we may reduce the problems to one-parameter ones by measuring all distances in units of the mean free path. This is not possible in spherical problems except of course in the case when the paths in the various media are all equal.

4.1. Critical Problems.

The plane configuration we shall consider here is a symmetric one, consisting of a center "plate" of half-thickness a, of which the right hand half is "located" in the interval (0,a), with $f = f_1$, $\sigma = \sigma_1$, and unknowns A_1 and B_1, and consisting further of a middle plate in (a,b) with $f = f_2$, $\sigma = \sigma_2$, and unknowns C_1 and D_1, and an outside plate in (b,∞) with $f = f_3$,

$\sigma = \sigma_3$, and unknowns E_1 and F_1. We denote the various media with roman numerals starting at the origin, we measure dimensions in cm and σ in 1/cm, and understand by a_1, b_1, etc., $a\sigma_1$, $b\sigma_1$, etc. In order to have a stable configuration in absence of sources, we require further that at least one of the inside media be multiplying ($f > 0$).

The corresponding spherical configuration consists of a core with radius a, a shell with radii a and b, and an outside shell with radii b and ∞.

Symmetry about the origin requires that the angular distribution at r = 0 be isotropic, which implies $\Psi_k(0)$ = Constant for k = 0, 0 for k = 1,2,...(n-1), which in turn implies that $A_1 = B_1$. Continuity at the boundaries requires $\Psi_k^{I}(a_1) = \Psi_k^{II}(a_2)$, and $\Psi_k^{II}(b_2) = \Psi_k^{III}(b_3)$. Finally scattering and/or capture in the outside medium implies $\Psi_k^{III}(\infty) = 0$, so that $E_1 = 0$.

We now have for plates and spheres respectively, using the matrix notation

$$(4.1.1) \quad \begin{aligned} C_n G_n K_n(\mu_i^{(1)}, a_1)[\bar{A}_1] &= C_n G_n K_n(\mu_i^{(2)}, a_2)[\bar{C}_1], \\ C_n G_n K_n(\mu_i^{(2)}, b_2)[\bar{C}_1] &= C_n G_n K_n(\mu_i^{(3)}, b_3)[\bar{E}_1], \end{aligned}$$

$$(4.1.2) \quad \begin{aligned} C_n T_n(a_1)\, Q_n(\mu_i^{(1)}, a_1)[\bar{A}_1] &= C_n T_n(a_2)\, Q_n(\mu_i^{(2)}, a_2)[\bar{C}_1], \\ C_n T_n(b_2)\, Q_n(\mu_i^{(2)}, b_2)[\bar{C}_1] &= C_n T_n(b_3)\, Q_n(\mu_i^{(3)}, b_3)[\bar{E}_1], \end{aligned}$$

where we take the hyp-form of the matrices in the first and second media and the exp-form in the third or outside medium. Eliminating the unknowns of the second medium, we obtain

$$(4.1.3)\qquad \begin{aligned} &K_n^*(\mu_1^{(1)},a_1)\,[\bar{A}_1]^* = K_n(\mu_1^{(2)},b_2-a_2)K_n^*(\mu_1^{(3)},b_3)\,[\bar{E}_1]^*,\\ &Q_n^*(\mu_1^{(1)},a_1)\,[\bar{A}_1]^* =\\ &\quad = T_n(a_1,a_2)Q_n(\mu_1^{(2)},b_2-a_2)T_n(b_2,b_3)Q_n^*(\mu_1^{(3)},b_3)\,[\bar{E}_1]^*,\end{aligned}$$

where the (*) serves to indicate that the dimensions have been reduced because of the physical conditions imposed. Thus, since $A_1 = B_1$ the last $\frac{1}{2}n$ elements in $[\bar{A}_1]$ equal zero, and this means that K_n^* and Q_n^* (for the first medium) have been reduced to n by $\frac{1}{2}n$ matrices consisting of the first $\frac{1}{2}n$ columns of K_n and Q_n respectively. Also, since $E_1 = 0$ the first $\frac{1}{2}n$ elements of $[\bar{E}_1]$ equal zero, so that K_n^* and Q_n^* (for the third medium) consist of the last $\frac{1}{2}n$ columns of K_n and Q_n (in exp-form) respectively.

The matrix equations (4.1.3) can be interpreted as two systems of linear equations each consisting of n equations in n unknowns. The following procedure enables us to reduce the order of these systems by a factor 2. We multiply left and right hand sides of (4.1.3) by $K_n^{-1*}(\mu_1^{(1)},a_1)$ consisting of the last $\frac{1}{2}n$ rows of $K_n^{-1}(\mu_1^{(1)},a_1)$, duplicate these steps in the spherical case using $Q_n^{-1*}(\mu_1^{(1)},a_1)$, and obtain

$$(4.1.4)\qquad \begin{aligned} &K_n^{-1*}(\mu_1^{(1)},a_1)K_n(\mu_1^{(2)},b_2-a_2)K_n^*(\mu_1^{(3)},b_3)\,[\bar{E}_1]^* = (0)_{\frac{1}{2}n}[\bar{A}_1]^*,\\ &Q_n^{-1*}(\mu_1^{(1)},a_1)T_n(a_1,a_2)Q_n(\mu_1^{(2)},b_2-a_2)T_n(b_2,b_3)Q_n^*(\mu_1^{(3)},b_3)\,[\bar{E}_1]^* =\\ &\quad = (0)_{\frac{1}{2}n}[\bar{A}_1]^*,\end{aligned}$$

where $(0)_n$ denotes a n by n matrix with zero elements.

If information about above configurations is given so that in effect we have but one unknown parameter, the latter (the critical number of the configuration) may be found from (4.1.4) since the solutions of theses systems of equations require that the determinants of the matrix factors on the left hand side vanish.

The formulas for the two-medium case may be obtained by letting b approach a in which case $K_n(\mu_1, b_2-a_2)$ and $Q_n(\mu_1, b_2-a_2)$ become unit matrices and $T_n(a_1,a_2)T_n(a_2,a_3)$ equals $T_n(a_1,a_3)$. The formulas for the N-medium case are readily obtained by inserting N-3 additional K_n's and Q_nT_n's before the last K_n and Q_n in (4.1.4), corresponding to the elimination of the unknowns of other central media. We note that the order of the matrix products in (4.1.4) is independent of the number of media and equal to $\frac{1}{2}n$.

Note 1. In chapters III and IV we have restricted the discussion to the case when n is even. The additional term corresponding to $\mu_{\frac{n+1}{2}} = 0$, which enters the expressions for $\Psi_k(r)$ in the odd case, can be interpreted to give a contribution to the moments at the boundaries but no contribution elsewhere. Now it is possible to show that for n odd Ψ_{n-1}-approximation is formally equivalent to Ψ_{n-2}-approximation. They will differ in detail, instead of the zeroes of $G_{n-1}(f,\mu)$ we use the zeroes of $G_n(f,\mu)$ excluding $\mu_{\frac{n+1}{2}}$, and instead of $T_{k+1}^{(n-1)}$-coefficients G_n and $T_n(r)$ will have elements involving the $T_{k+1}^{(n)}$ - coefficients. The coefficient C_nG_n will cancel in the plane as well as the spherical case; in the latter case because $T_n(r)$ may be written $G_nT_n(0,r)$. The equivalence of ψ_{n-1}-approximation and ψ_{n-2}-approximation now follows from the fact that the column in $K_n(\mu_i,r)$, $T_n(0,r)$, and $Q_n(\mu_i,r)$ corresponding to $\mu_{\frac{n+1}{2}}$, contains zero elements for row indices 1 through n-1.

Note 2. If in the plane case we have $\sigma = 0$ in one of the interior media, we may because of our observations in paragraph (3.3) simply let it collapse. For instance, if the middle medium has $\sigma = 0$, our boundary condition becomes $\psi_k(a_1) = \psi_k(b_3)$. If the outside medium (in either the plane or spherical case) has a zero "cross-section", we may assign to it arbitrary parameters, e.g., $f = -1$ and $\sigma = \infty$, as long as we characterize it in a manner that will permit no neutrons to return to the interior media.

If the core of a spherical configuration has zero crossection we use the matrix representation developed in paragraph (3.3). The condition that the density be finite at $r = 0$ implies $A_i = 0$ for even indices. We then have corresponding to (4.1.4)

$$(4.1.5) \qquad E_{\frac{1}{2}n}T_n(a_2)Q_n(\mu_1^{(2)},b_2-a_2)T_n(b_2,b_3)Q_n^*(\mu_1^{(3)},b_3)[E_1]^* = \\ = (0)_{\frac{1}{2}n}[A_1]^*,$$

where

$$(4.1.6) \qquad E_{\frac{1}{2}n} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

If the inside shell has a crossection σ equal to zero we obtain, replacing (4.1.4),

$$(4.1.7) \qquad Q_n^{-1*}(\mu_1^{(1)}, a_1) T_n(a_1) D_n(a,b) T_n(b_3) Q_n^{*}(\mu_1^{(3)}, b_3) [E_1]^* =$$
$$= (0)_{\frac{1}{2}n} [A_1]^* .$$

Note 3. If $f = 0$ we may, instead of using the special matrices developed in paragraph (3.4) factor out a diagonal matrix with elements μ_1^{n-1} from $K_n(\mu_1, r)$ and $Q_n(\mu_1, r)$ and include it with the associated vector. We may then let μ_1 approach zero and introduce A_0 and B_0 as defined in the beginning of paragraph (3.4).

4.2. Transmission Problems.

The techniques required in order to obtain the density distribution, $\Psi_0(r)$, and quantities associated with it will be developed here in connection with problems where we are primarily interested in the current transmission. The current at a point r by definition is given by

$$(4.2.1) \qquad \text{Current} = \int_{-1}^{1} \mu \Psi(r,u)d\mu = \Psi_1(r),$$

and the current transmission across a medium in (a,b) by

$$(4.2.2) \qquad \text{Transmission} = \Psi_1(b)/\Psi_1(a) \times S,$$

where S is a surface factor equal to unity in the plane case and b^2/a^2 in the spherical case.

The plane configuration which we consider in this paragraph consists of an inside semi-infinite plate in $(-\infty,a)$ with $f = f_1$ and $\sigma = \sigma_1$, a middle plate in (a,b), $f = f_2$, $\sigma = \sigma_2$, $a < b$, and an outside semi-infinite plate in (b,∞). We take b = 0 in this case, and normalize by requiring that $\Psi_1(b) = 1$. The corresponding spherical system consists of a core in (0,a), a shell in (a,b), and a semi-infinite shell in (b,∞). Here as in the plane case we take $\Psi_1(b) = 1$. Further we assume that $-1 \leq f_1, f_2, f_3 \leq 0$, and that neutrons are supplied by a plane source (shell source) at ∞.

In these problems we number the media and the parameters, and denote the unknowns of each medium, as in paragraph (4.1). The boundary conditions are again those of the previous paragraph, i.e., the continuity of the moments. Further, we require in the plane case that $\Psi_k(-\infty) = 0$, so that $B_i = 0$, and in the spherical case that the angular distribution at $r = 0$ be isotropic, so that $A_i = B_i$. The terms in $\Psi_k(r)$ involving μ_i, $i > 1$, can be interpreted as the contribution to the asymptotic terms of $\Psi_k(r)$, those terms involving μ_1, near a boundary. The condition in the third medium is, therefore, that the non-asymptotic part of the density vanishes, i.e., $E_i = 0$, $i > 1$. We then have

$$(4.2.3)\qquad \begin{aligned} C_n G_n K_n^*(\mu_i^{(1)}, a_1)[\bar{A}_i]^* &= C_n G_n K_n(\mu_i^{(2)}, a_2)[\bar{C}_i] \,, \\ C_n G_n K_n(\mu_i^{(2)}, b_2)[\bar{C}_i] &= C_n G_n K_n^*(\mu_i^{(3)}, b_3)[\bar{E}_i]^* \,, \end{aligned}$$

in the plane case, the matrices taken in exp-form, and

$$(4.2.4)\qquad \begin{aligned} C_n T_n(a_1) Q_n^*(\mu_i^{(1)}, a_1)[\bar{A}_i]^* &= C_n T_n(a_2) Q_n(\mu_i^{(2)}, a_2)[\bar{C}_i] \,, \\ C_n T_n(b_2) Q_n(\mu_i^{(2)}, b_2)[\bar{C}_i] &= C_n T_n(b_3) Q_n^*(\mu_i^{(3)}, b_3)[\bar{E}_i]^* \,, \end{aligned}$$

in the spherical case, the matrices associated with the first medium taken in hyp-form and the others in exp-form. Eliminating the constants in the middle medium we have

$$(4.2.5)\qquad K_n^*(\mu_i^{(1)}, a_1)[\bar{A}_i]^* = K_n(\mu_i^{(2)}, b_2 - a_2) K_n^*(\mu_i^{(3)}, b_3)[\bar{E}_i]^* \,,$$

$$(4.2.6)\qquad Q_n^*(\mu_i^{(1)}, a_1)[\bar{A}_i]^* = T_n(a_1, a_2) Q_n(\mu_i^{(2)}, b_2 - a_2) T_n(b_2, b_3) \cdot Q_n^*(\mu_i^{(3)}, b_3)[\bar{E}_i]^* \,,$$

where

$K_n^*(\mu_i^{(1)}, a_1)$ is comprised of the first $\frac{1}{2}n$ coulumns of K_n given in exp-form, $Q_n^*(\mu_i^{(1)}, a_1)$ of the first $\frac{1}{2}n$ columns of Q_n given in hyp-form; $K_n^*(\mu_i^{(3)}, b_3)$ of the first $\frac{1}{2}n+1$ columns of K_n given in exp-form, and $Q_n^*(\mu_i^{(3)}, b_3)$ of the same part of Q_n. Proceeding as from (4.1.3) to (4.1.4) we obtain

$$(4.2.7)\quad \begin{aligned} &K_n^{-1*}(\mu_i^{(1)}, a_1)K_n(\mu_i^{(2)}, b_2-a_2)K_n^*(\mu_i^{(3)}, b_3)[\bar{E}_i]^* = (0)_{\frac{1}{2}n}[\bar{A}_i]^*, \\ &Q_n^{-1*}(\mu_i^{(1)}, a_1)T_n(a_1, a_2)Q_n(\mu_i^{(2)}, b_2-a_2)T_n(b_2, b_3)Q_n^*(\mu_i^{(3)}, b_3)[\bar{E}_i]^* = \\ &\qquad = (0)_{\frac{1}{2}n}[\bar{A}_i]^*, \end{aligned}$$

and reversing the elimination

$$(4.2.8)\quad \begin{aligned} &K_n^{-1*}(\mu_i^{(3)}, b_3)K_n(\mu_i^{(2)}, a_2-b_2)K_n^*(\mu_i^{(1)}, a_1)[\bar{A}_i]^* = (1)_{\frac{1}{2}n+1}[\bar{E}_i]^*, \\ &Q_n^{-1*}(\mu_i^{(3)}, b_3)T_n(b_3, b_2)Q_n(\mu_i^{(2)}, a_2-b_2)T_n(a_2, a_1)Q_n^*(\mu_i^{(1)}, a_1)[\bar{A}_i]^* = \\ &\qquad = (1)_{\frac{1}{2}n+1}[\bar{E}_i]^*, \end{aligned}$$

where

$$(1)_{\frac{1}{2}n+1} = \begin{pmatrix} 0 & 0 & \cdots\cdots & 0 & 1 \\ 0 & 0 & \cdots\cdots & 0 & 0 \\ \cdots & \cdots & \cdots\cdots & \cdots & \cdots \end{pmatrix}.$$

Formulas (4.2.7) represent systems of $\frac{1}{2}n$ equations in $\frac{1}{2}n+1$ unknowns. We may then solve for $\bar{E}_i/\bar{E}_{\frac{1}{2}n+1}$, $i = 1,2,\ldots\frac{1}{2}n$. Similarly we may solve (4.2.8) for $\bar{A}_i/\bar{E}_{\frac{1}{2}n+1}$. Then substituting the values obtained for these unknowns in the first (or second) part of (4.2.3) and (4.2.4), we may solve for $\bar{C}_i/\bar{E}_{\frac{1}{2}n+1}$. Finally $\bar{E}_{\frac{1}{2}n+1}$ may be found by making use of the assumption $\Psi_1(b) = 1$. To obtain the transmission we need not bother about the last step, since this quantity is a ratio.

The same procedure can of course be employed to find the density distribution in a critical configuration after that the critical number has been found. We have merely to decide upon some normalization of the density.

4.3. Problems Involving Neutron Sources.

In this paragraph we shall consider a few simple configurations containing a source of neutrons, a symmetrical plane system with a unit plane source at $r = 0$, giving rise to a distribution

$$(4.3.1) \qquad p(r) = \tfrac{1}{2}E_1(|r|)$$

of primary neutrons (C. Mark 2, p. 25), and a spherical system with a unit point source at $r = 0$, giving rise to a distribution

$$(4.3.2) \qquad p(r) = \frac{\sigma^2}{4\pi r^2}\, e^{-r}$$

of primary neutrons. The secondary neutrons in the system can then be regarded as having been generated by the isotropic source functions $q(r) = (1+f)p(r)$.

A unit plane source at $r = a$ will give rise to $p(r) = \tfrac{1}{2}E_1(|r-a|)$. A unit point source may be regarded as the limiting case of a unit shell source at $r = a$, which gives rise to the distribution of primaries: $p(r) =$ $= (1/8\pi ar)\left[E_1(|r-a|)-E_1(|r+a|)\right]$. The function $E_1(x)$ is the exponential integral function sometimes denoted by $-Ei(-x)$. See G. Placzek 2.

Considering the geometry of above configurations we find that $\Psi_k^s(r)$, the angular moments for the secondary neutrons, vanish for odd k at $r = 0$. Hence $\Psi_k(0)$ is given by $\psi_k^p(0)$, the moments for the primary neutrons, for odd k. Going back to the Spherical Harmonics form of the Transport Equation, (3.1.2) and (3.2.2), we obtain, letting $\psi_0^p(r) = p(r)$,

$$\psi_k^p(r) = \tfrac{1}{2}\int_{-1}^{1} P_k(\mu) e^{-r/\mu} \frac{d\mu}{\mu} , \tag{4.3.3}$$

$$r^{k+1}\psi_k^p(r) = \frac{\sigma^2 r^{k+1}}{4\pi r^2} e^{-r} . \tag{4.3.4}$$

As r approaches zero we then have

$$\psi_1^p(0) = \tfrac{1}{2}, \quad \psi_3^p(0) = -1/3, \quad \psi_k^p(0) = \tfrac{1}{2}(-1)^{\frac{k-1}{2}} \frac{2 \cdot 4 \cdots (k-1)}{3 \cdot 5 \cdots k} , \quad k = 1, 3, \ldots \tag{4.3.5}$$

$$r^2\psi_1^p(0) = 1/4\pi, \quad r^{k+1}\psi_k^p(0) = 0, \quad k = 3, 5, \ldots \tag{4.3.6}$$

If the sources are surrounded by a non-scattering plate and core respectively the situation is similar to the one described above. Again we have $\psi_k^s(a) = 0$, k odd, where a is the coordinate of the outside boundary of the non-scattering medium. The boundary conditions at r = a are then the relations (4.3.3) and (4.3.4) evaluated at r = a, k odd.

Above conditions are sufficient in order to investigate configurations with sources with the SH-method and the matrix techniques developed in paragraphs (4.1) and (4.2). However, the angular distribution near a source is clearly quite unisotropic. We can , therefore, expect much better results if we apply the SH-method to the secondary neutrons.

We start with the differential systems

$$(4.3.7)\quad (k+1)D_r\Psi_{k+1}(r)+kD_r\Psi_{k-1}(r)+(2k+1)\Psi_k(r) = \\ = \begin{cases} (1+f)\Psi_0(r)+q(r), & k = 0, \\ 0, & k = 1,2,\ldots(n-1), \end{cases}$$

$$(4.3.8)\quad (k+1)\left[D_r+\frac{k+2}{r}\right]\Psi_{k+1}(r)+k\left[D_r-\frac{k-1}{r}\right]\Psi_{k-1}(r)+(2k+1)\Psi_k(r) = \\ = \begin{cases} (1+f)\Psi_0(r)+q(r), & k = 0, \\ 0, & k = 1,2,\ldots(n-1), \end{cases}$$

and the solutions of these are obtained as the sum of the general solution of (3.1.2) and (3.2.2), and a particular solution of (4.3.7) and (4.3.8). Using the recurrence formula for $G_n(f,\mu)$ and formulas (3.1.20) and (3.1.21) we verify readily that the following expressions are the general solutions of (4.3.7) and (4.3.8):

$$(4.3.9)\quad \Psi_k(r) = \sum_{i=1}^{n/2} G_k(f,\mu_i)\Big\{(-1)^k\Big[A_i+\tfrac{1}{2}\mu_i c_i\int^r q(r')e^{-r'/\mu_i}dr'\Big]e^{r/\mu_i}+ \\ +\Big[B_i-\tfrac{1}{2}\mu_i c_i\int^r q(r')e^{r'/\mu_i}dr'\Big]e^{-r/\mu_i}\Big\},$$

$$(4.3.10)\quad \Psi_k(r) = \sum_{i=1}^{n/2} G_k(f,\mu_i)\Big\{(-1)^k\Big[A_i+\tfrac{1}{2}\mu_i c_i\int^r q(r')\frac{r'}{\mu_i}e^{-r'/\mu_i}dr'\Big]H_k(r/\mu_i)+ \\ +\Big[B_i-\tfrac{1}{2}\mu_i c_i\int^r q(r')\frac{r'}{\mu_i}e^{r'/\mu_i}dr'\Big]H_k(-r/\mu_i).$$

In the cases we are considering we have

$$(4.3.11)\quad \int^r q(r')e^{-r'/\mu_i}dr' = \tfrac{1}{2}(1+f)\int^r E_1(r')e^{-r'/\mu_i}dr' = \\ = \tfrac{1}{2}(1+f)\mu_i\Big\{E_1[r(1+1/\mu_i)]-E_1(r)e^{-r/\mu_i}\Big\},$$

$$(4.3.12)\quad \int^r q(r')e^{r'/\mu_i}dr' = -\tfrac{1}{2}(1+f)\mu_i\Big\{E_1[r(1-1/\mu_i)]-E_1(r)e^{r/\mu_i}\Big\},$$

where $E_1(-x)$, x positive, equals $-Ei(x)$, and $c_i = D_f(1/\mu_i^2)$; and in the spherical case

$$(4.3.13)\quad \int^{r} q(r')\frac{r'}{\mu_1} e^{-r'/\mu_1} dr' = \frac{\sigma^2(1+f)}{4\pi}\int^{r}\frac{1}{r'\mu_1} e^{-r'/\mu_1 - r'} dr' = -\frac{\sigma^2(1+f)}{4\pi\mu_1} E_1[r(1+1/\mu_1)],$$

$$(4.3.14)\quad \int^{r} q(r')\frac{r'}{\mu_1} e^{r'/\mu_1} dr' = \frac{\sigma^2(1+f)}{4\pi\mu_1} E_1[r(1-1/\mu_1)] .$$

Substituting (4.3.13) and (4.3.14) in (4.3.9) and (4.3.10) we obtain, using (3.1.20),

$$(4.3.15)\quad \Psi_k(r) = \sum_{i=1}^{n/2} G_k(f,\mu_1)\Big\{(-1)^k\big[A_1+\tfrac{1}{4}(1+f)\mu_1^2 c_1 E_1(r\overline{1+1/\mu_1})\big]e^{r/\mu_1} + \big[B_1+\tfrac{1}{4}(1+f)\mu_1^2 c_1 E_1(r\overline{1-1/u_1})\big]e^{-r/\mu_1} - \begin{cases}\tfrac{1}{2}(1+f)/f\cdot E_1(r), & k=0,\\ 0, & k=1,2,\dots(n-1),\end{cases}$$

$$(4.3.16)\quad \Psi_k(r) = \sum_{i=1}^{n/2} G_k(f,\mu_1)\Big\{(-1)^k\big[A_1-\tfrac{1}{8\pi}(1+f)c_1E_1(r\overline{1+1/u_1})\big]H_k(r/\mu_1) + \big[B_1-\tfrac{1}{8\pi}(1+f)c_1E_1(r\overline{1-1/\mu_1})\big]H_k(-r/\mu_1) ,$$

where the bar serves as parantheses.

We may also write the above expressions in hyp-form, obtaining, with $\mathcal{E}_k(r,\mu_1) = [E_1(r\overline{1+1/\mu_1}) + (-1)^k E_1(r\overline{1-1/\mu_1})]$,

$$(4.3.17)\quad \Psi_k(r) = (-1)^k\sum_{i=1}^{n/2} G_k(f,\mu_1)\Big\{\big[A_1+(-1)^kB_1+\tfrac{1}{2}(1+f)\mu_1^2c_1\mathcal{E}_k(r,\mu_1)\big]\cosh r/\mu_1 + \big[A_1+(-1)^{k+1}B_1+\tfrac{1}{2}(1+f)\mu_1^2c_1\mathcal{E}_{k+1}(r,\mu_1)\big]\sinh r/\mu_1 - \begin{cases}\tfrac{1}{2}(1+f)/f\cdot E_1(r), & k=0,\\ 0, & k=1,2,\dots(n-1),\end{cases}$$

$$(4.3.18)\quad \Psi_k(r) = (-1)^k\sum_{i=1}^{n/2} G_k(f,\mu_1)\Big\{\big[A_1+(-1)^kB_1-\tfrac{1+f}{8\pi}c_1\mathcal{E}_k(r,\mu_1)\big]H_k^{(1)}(r/\mu_1)\sinh r/\mu_1 + \big[A_1+(-1)^{k+1}B_1-\tfrac{1+f}{8\pi}c_1\mathcal{E}_{k+1}(r,\mu_1)\big]H_k^{(2)}(r/\mu_1)\cosh r/\mu_1 .$$

We may now regard the inside brackets in the last four formulas above as new unknowns. Then the matrix technique of the preceding paragraphs is applicable, except in a middle medium where an additional vector is introduced at the second boundary, to keep the new unknowns equal at the two boundaries of the middle medium.

It must be noted that in the second medium above formulas will change slightly due to the fact that in that medium $p(r) = \frac{1}{2}E_1[r-(a\sigma_2-a\sigma_1)]$ for plates and $p(r) = \frac{\sigma_2^2}{4\pi r^2} e^{-r+(a\sigma_2-a\sigma_1)}$ for spheres, so that $p(a\sigma_1) = p(a\sigma_2)$. In the third medium we will have still other expressions for $p(r)$. For integrals involving the E_1-function see J. LeCaine 1.

The condition on the odd moments at $r = 0$ gives the following expressions for $A_i - B_i$, due to the fact that the determinant $G_{2i-1}(f, \mu_i)$ does not vanish:

$$(4.3.19) \qquad A_i - B_i = \tfrac{1}{4}(1+f)c_i \log\left|\frac{1+1/\mu_i}{1-1/\mu_i}\right|,$$

$$(4.3.20) \qquad A_i - B_i = -\frac{1+f}{8\pi} c_i \log\left|\frac{1+1/\mu_i}{1-1/\mu_i}\right|.$$

See C. Mark 2, pp.

<u>Plane source in infinite medium.</u>

If a plane source at $r = 0$ is surrounded by an infinite medium characterized by the capture probability f, $-1 < f \leq 0$, we have $A_i = 0$ and hence the following formulas for the angular moments, including the primary neutrons:

$$(4.3.21) \qquad \psi_0(r) = -(1/2f)E_1(r) - \tfrac{1}{4}(1+f)\sum_{i=1}^{n/2}\mu_i^2 c_i \Big[e^{-r/\mu_i}\log\left|\frac{1+1/\mu_i}{1-1/\mu_i}\right| - E_1(r\overline{1+1/\mu_i})e^{r/\mu_i} - E_1(r\overline{1-1/\mu_i})e^{-r/\mu_i}\Big]$$

and

$$\Psi_k(r) = \frac{1}{2}\int_0^1 \frac{e^{-r/\mu}}{\mu} P_k(\mu)d\mu - \frac{1}{4}(1+f)\sum_{i=1}^{n/2} \mu_i^2 G_k(f,\mu_i)c_i \cdot$$

(4.3.22)

$$\cdot \left[e^{r/\mu_i}\log\left|\frac{1+1/\mu_i}{1-1/\mu_i}\right| - (-1)^k \cdot e^{r/\mu_i} E_1(r\overline{1+1/\mu_i}) - e^{-r/\mu_i} E(r\overline{1-1/\mu_i})\right].$$

Point source in an infinite medium.

For the angular moments in the case of a point source in an infinite medium we have the following expressions, corresponding to (4.3.21) and (4.3.22):

$$\Psi_0(r) = \frac{\sigma^2}{4\pi r^2} e^{-r} + \frac{1+f}{8\pi}\sum_{i=1}^{n/2} c_i \left[H_0(-r/\mu_i)\log\left|\frac{1+1/\mu_i}{1-1/\mu_i}\right| - \right.$$

(4.3.23)

$$\left. -H_0(r/\mu_i)E_1(r\overline{1+1/\mu_i}) - H_0(-r/\mu_i)E_1(r\overline{1-1/\mu_i})\right],$$

$$\Psi_k(r) = \frac{\sigma^2}{4\pi r^2} e^{-r} + \frac{1+f}{8\pi}\sum_{i=1}^{n/2} G_k(f,\mu_i)c_i \left[H_k(-r/\mu_i)\log\left|\frac{1+1/\mu_i}{1-1/\mu_i}\right| - \right.$$

(4.3.24)

$$\left. -(-1)^k H_k(r/\mu_i)E_1(r\overline{1+1/\mu_i}) - H_k(-r/\mu_i)E_1(r\overline{1-1/\mu_i})\right.$$

We note that (4.3.23) is the derivative owith respect to r of (4.3.21), multiplied by $-1/2\pi r$.

CHAPTER V

TWO-MEDIUM PROBLEMS, EXACT SOLUTIONS.

5.1. Critical Problems.

In the two-medium case the matrix products of the previous chapter simplify considerably, especially if we restrict ourselves to plates and to those spherical configurations where $\sigma_1 = \sigma_2$. Furthermore, in those problems - and perhaps in others - it becomes possible to pass to the limit as n approaches infinity, if we make a simplifying assumption regarding the behavior of the density near boundaries.

From (4.1.4) we have for plates and spheres respectivly

$$K_n^{-1*}(\mu_i, a_1) K_n^*(\nu_i, a_2) \left[\bar{C}_i\right] = 0, \tag{5.1.1}$$

$$Q_n^{-1*}(\mu_i, a_1) Q_n^*(\nu_i, a_1) \left[\bar{C}_i\right] = 0, \tag{5.1.2}$$

where μ_i are the roots of $G_n(f_1, \mu)$ and ν_i the corresponding roots for $f = f_2$. Multiplying the matrices we obtain

$$\left(\frac{\nu_j th_i + \mu_i}{\nu_j^2 - \mu_i^2}\right) \left[\frac{\nu_i D_i e_i}{S_n(\nu_i)} \prod_{j=1}^{n/2} (\nu_i^2 - \mu_j^2)\right] = 0, \tag{5.1.3}$$

$$\left(\frac{\nu_j cth_i + \mu_i}{\nu_j^2 - \mu_i^2}\right) \left[-\frac{\nu_i^2 D_i e_i}{S_n(\nu_i)} \prod_{j=1}^{n/2} (\nu_i^2 - \mu_j^2)\right] = 0, \tag{5.1.4}$$

where we have assumed that n is even in order to simplify the notation.

Using formulas (2.3.1) and (2.3.5) and performing some straightforward algebraic operations we have

(5.1.5) $$G_n(f_1,\nu_i) = \text{Const.}\left[f_1 S_n(\nu_i)+R_{n-2}(\nu_i)\right] = \text{Const.}\left[(f_1-f_2)S_n(\nu_i)+f_2 S_n(\nu_i)+R_{n-2}(\nu_i)\right].$$

On the other hand we have

(5.1.6) $$G_n(f_1,\nu_i) = \text{Const.}\left[f_1 \prod_{j=1}^{n/2}(\nu_i^2-\mu_j^2)\right].$$

Hence

(5.1.7) $$S_n(\nu_i) = \frac{f_1}{f_1-f_2}\prod_{j=1}^{n/2}(\nu_i^2-\mu_j^2),$$

(5.1.8) $$S_n(\mu_i) = \frac{f_2}{f_2-f_1}\prod_{j=1}^{n/2}(\mu_i^2-\nu_j^2).$$

Formulas (5.1.3) and (5.1.4) may then be rewritten in the following form:

(5.1.9) $$\left(\frac{\nu_j th_i+\mu_i}{\nu_j^2-\mu_i^2}\right)\left[\frac{f_1-f_2}{f_1}\,\nu_i D_i e_i\right],$$

(5.1.10) $$\left(\frac{\nu_j cth_i+\mu_i}{\nu_j^2-u_i^2}\right)\left[-\frac{f_1-f_2}{f_1}\,\nu_i^2 D_i e_i\right].$$

Similarly, eliminating the unknowns of the infinite shell, we obtain

(5.1.11) $$\left(\frac{\nu_i th_j+\mu_j}{\nu_i^2-\mu_j^2}\right)\left[\frac{f_2-f_1}{f_2}\,\mu_i ch_i(A_i+B_i)\right],$$

(5.1.12) $$\left(\frac{\nu_i cth_j+\mu_j}{\nu_i^2-\mu_j^2}\right)\left[\frac{f_2-f_1}{f_2}\,\mu_i^2 sh_i(A_i+B_i)\right],$$

where $th_i = \tanh a_1/\mu_i$, etc., and $e_i = e^{-a\sigma_i/\nu_i}$.

The critical number is then the value of the unknown parameter which makes the following determinants vanish:

$$\left|\left(\frac{\nu_j th_i+\mu_i}{\nu_j^2-\mu_i^2}\right)\right| = 0, \text{ plane case,} \quad (5.1.13)$$

$$\left|\left(\frac{\nu_j cth_i+\mu_i}{\nu_j^2-\mu_i^2}\right)\right| = 0, \text{ spherical case.} \quad (5.1.14)$$

Above determinants involve three parameters, a_1, f_1, and f_2.

Once the critical number has been found and some normalization introduced, e.g., $\psi_o(0) = 1$, we may solve for the unknowns and calculate $\psi_o(r)$ for as many values of r as we desire.

5.2. Critical Problems, the End-Point Method.

The terms in $\psi_o(r)$ involving μ_i, $i > 1$, may be interpreted as boundary corrections to the asymptotic density (the terms in $\psi_o(r)$ involving μ_1), the magnitude of which decreases rapidly as the distance from the boundary in question increases. We now assume that the boundaries in our configurations are well separated, in fact, to such an extent that we may regard them as infinitely separated. In the two-medium problems now under consideration this assumption implies th_i and cth_i equal to unity for $i > 1$. To find the critical number of this simplified model (applying the "end-point method") amounts to obtaining the value of the unassigned parameter which satisfies

$$(5.2.1) \qquad \left| \left(\frac{\nu_j th_i + \mu_i}{\nu_j^2 - \mu_i^2} \right) \right| = 0, \quad th_i = 1 \text{ for } i \geq 1,$$

$$(5.2.2) \qquad \left| \left(\frac{\nu_j cth_i + \mu_i}{\nu_j^2 - \mu_i^2} \right) \right| = 0, \quad cth_i = 1 \text{ for } i \geq 1.$$

We observe that (5.2.2) is a linear function in cth_1. Denoting (5.2.2) by $F(cth_1)$ and using the two-point formula for linear functions we have

$$(5.2.3) \qquad F(cth_1) \equiv \frac{F(1)-F(-1)}{1-(-1)}\{cth_1-(-1)\} + F(-1) = 0,$$

and hence

$$(5.2.4) \qquad cth_1 = \frac{F(-1)+F(1)}{F(-1)-F(1)}, \quad \text{and}$$

$$(5.2.5) \qquad a_1 = \mu_1 \operatorname{arcoth} \frac{F(-1)+F(1)}{F(-1)-F(1)} = \tfrac{1}{2}\mu_1 \log \frac{F(-1)}{F(1)} .$$

The determinants $F(1)$ and $F(-1)$ are readily obtained by factoring.

$$(5.2.6) \qquad F(1) = \frac{\prod(\nu_j - \nu_i)\prod(\mu_j - \mu_i)\prod(\mu_j - \mu_i)}{\prod(\nu_j - \mu_i)\prod(\nu_j - \mu_i)},$$

$$(5.2.7) \qquad F(-1) = \frac{\prod\limits_{j>i\geq 1}^{n/2}(\nu_j - \nu_i)\prod\limits_{j\geq 1}^{n/2}(\mu_j + \mu_i)\prod\limits_{j>i>1}^{n/2}(\mu_j - \mu_i)}{\prod\limits_{j\geq 1}^{n/2}(\nu_j + \mu_i)\prod\limits_{\substack{i>1\\ j\geq 1}}^{n/2}(\nu_j - \mu_i)},$$

where the products in (5.2.6) are taken over the same indices as in (5.2.7).

From (5.2.6) and (5.2.7) follows that

(5.2.8) $$\frac{F(-1)}{F(1)} = \frac{\prod_{j>1}^{n/2}(\mu_j+\mu_1)\prod_{j\geq 1}^{n/2}(\nu_j-\mu_1)}{\prod_{j>1}^{n/2}(\mu_j-\mu_1)\prod_{j\geq 1}^{n/2}(\nu_j+\mu_1)} ,$$

and

(5.2.9) $$a_1 = |\mu_1| \operatorname{arcot} -|\mu_1|/\nu_1 + |\mu_1|\sum_{j=2}^{n/2}\left[\arctan |\mu_1|/\nu_j - \arctan |\mu_1|/\mu_j\right] .$$

Using the formula

(5.2.10) $$\arctan A - \arctan B = \arctan \frac{A-B}{1+AB} ,$$

we obtain the following expression for a_1 equivalent to (5.2.9):

(5.2.11) $$a_1 = |\mu_1| \operatorname{arcot} -|\mu_1|/\nu_1 + |\mu_1|\sum_{j=2}^{n/2}\arctan \frac{(\mu_j-\nu_j)}{|\mu_1|(1+\mu_j\nu_j/|\mu_1|^2)} .$$

5.3. The End-Point Method, Exact Solution.

As n approaches ∞, $\mu_1 \to u_0$, $\nu_1 \to \nu_0$, and $\mu_j - \nu_j \to 0$. In the limit, therefore, we have

(5.3.1) $$a_1 = |\mu_0| \operatorname{arcot} -|\mu_0|/\nu_0 + \lim_{n\to\infty}\sum_{j=2}^{n/2}\frac{(\mu_j-\nu_j)}{1 + \nu_j^2/|\mu_0|^2} ,$$

and applying Theorem 2.7.3

(5.3.2) $$a_1 = |\mu_0| \operatorname{arcot} -|\mu_0|/\nu_0 + (1/\pi)\int_0^1\left[\arctan g(f_2,\mu) - \arctan g(f_1,\mu)\right]\frac{d\mu}{1 + \mu^2/|\mu_0|^2} .$$

This result has also been obtained by "integral theory" methods. See Frankel and Goldberg 1, p. 51, where the second term of (5.3.2), denoted by Δx, is given. Frankel and Goldberg denote arctan $g(f,\mu)$ by T_c, μ_o by $1/k$, $1+f$ by c, and use primes to distinguish quantities associated with the outside medium from those of the core. In what follows we denote the second term of (5.3.2) by Δa_1.

Theorem 2.7.3 does in fact establish a link between the integral theory used in Frankel and Goldberg and the Spherical Harmonics Method. There is no need for repeating the analysis in that report in its entirety using the methods of this report. We shall, however, in order to illustrate the technique developed here, derive the formulas for the neutron density.

Formula (5.1.10) represents a system of $\frac{1}{2}n$ linear simultaneous equations. From the last $\frac{1}{2}n-1$ of these, remembering that in these equations $cth_i = 1$, we have

$$(5.3.3) \qquad \frac{\prod_{j=1}^{n/2}(\nu_i-\mu_j)}{\nu_i^2\prod_{j\neq i}(\nu_i-\nu_j)}\,\frac{1}{(\nu_i-\mu_1)D_1 e_i} = C, \quad i = 1,2,\dots\tfrac{1}{2}n,$$

where C is a factor of proportionality independent of i. We rewrite formula (5.3.3) as follows:

$$(5.3.4) \qquad \frac{\prod_{j=1}^{n/2}(\nu_i^2-\mu_j^2)}{\nu_i^2\prod_{j\neq i}(\nu_i^2-\nu_j^2)}\,\frac{\prod_{j=2}^{n/2}(\nu_i+\nu_j)}{\prod_{j=2}^{n/2}(\nu_i+\mu_j)}\,\frac{(\nu_1+\nu_i)}{(\mu_1+\nu_i)}\,\frac{1/\nu_i}{2(\mu_1-\nu_i)D_1 e_i} = -C,$$

and make use of the following expression for $\nu_i^2 c(\nu_i)$, where $c(\nu_i) = \frac{d}{df_2}(1/\nu_i^2)$, which is a direct consequence of (5.1.7) and (3.1.18),

$$(5.3.5) \qquad \nu_i^2 c(\nu_i) = \frac{f_1}{f_2(f_1-f_2)}\,\frac{\prod_{j=1}^{n/2}(\nu_i^2-\mu_j^2)}{\nu_i^2\prod_{j\neq i}(\nu_i^2-\nu_j^2)} \;.$$

Substituting (5.3.5) in (5.3.4) we obtain

$$(5.3.6)\qquad \frac{\prod_{j=2}^{n/2}(\nu_1+\nu_j)}{\prod_{j=2}^{n/2}(\nu_1+\mu_j)}\;\frac{(\nu_1+\nu_1)}{(\mu_1^2-\nu_1^2)}\;\frac{\nu_1 c(\nu_1)}{2D_1 e_i} = \frac{f_1 C}{f_2(f_2-f_1)}$$

which may also be written

$$(5.3.7)\qquad D_1 e_i = \frac{f_2(f_2-f_1)}{2f_1 C}\,\frac{(\nu_1+\nu_1)}{(\mu_1^2-\nu_1^2)}\,\nu_1 c(\nu_1)\, e^{\sum_{j=2}^{n/2}\log\left(1-\frac{\mu_j-\nu_j}{\nu_1+\mu_j}\right)}.$$

From (3.2.3) we have

$$(5.3.8)\qquad \Psi_0(r) = -\sum_{i=1}^{n/2} D_1 e_i\, \nu_1 (1/r) e^{(a_1-r)/\nu_1},$$

so that

$$(5.3.9)\qquad \Psi_0(r) = \frac{f_2(f_1-f_2)}{2f_1 Cr}\cdot \sum_{i=1}^{n/2}\frac{(\nu_1+\nu_1)}{(\mu_1^2-\nu_1^2)}\,\nu_1^2 c(\nu_1)\, e^{(a_1-r)/\nu_1+\sum_{j=2}^{n/2}\log\left(1-\frac{\mu_j-\nu_j}{\nu_1+\mu_j}\right)}.$$

Applying Theorem 2.7.3 and Theorem 2.7.4) we obtain in the limit as n approaches ∞

$$(5.3.10)\qquad \Psi_0(r) = A(f_1,f_2)(1/r)e^{-(r-a_1)/\nu_0 - v(\nu_0)} + B(f_1,f_2)(1/r)\int_0^1 \frac{d\nu}{w(f_2,\nu)}\,\frac{(\nu_0+\nu)}{(\mu_0^2-\nu^2)}\cdot \exp\left[-(r-a_1)/\nu - v(\nu)\right],$$

where

$$(5.3.11)\qquad \begin{aligned} v(\nu) &= (1/\pi)\int_0^1 \frac{d\mu}{\nu+\mu}\left[\arctan g(f_2,\mu) - \arctan g(f_1,\mu)\right],\\ w(f,\mu) &= \left\{\left[1-(1+f)\mu\,\mathrm{arcth}\,\mu\right]^2 + (\pi^2/4)(1+f)^2\mu^2\right\}, \end{aligned}$$

and

$$(5.3.12)\qquad A(f_1,f_2) = B(f_1,f_2)\left[4(\nu_0-1)/\nu_0(1+f_2)(1+f_2\nu_0^2)(\mu_0^2-\nu_0^2)\right],$$

$$(5.3.12)\qquad B(f_1,f_2) = f_2(f_2-f_1)/2f_1 C\ .$$

We can now compare the second term in (5.3.10) for instance with the corresponding result obtained by Frankel and Goldberg (1, p. 53, last entry in the fourth row). Except for a constant factor (depending on normalization) and different notation these two results are identical. The numerical work involved in tabulating the density using (5.3.10) is of course considerable.

The plane result corresponding to (5.3.2) differs from (5.3.2) in that arcot is replaced by arctan. And the plane result corresponding to (5.3.10) differs in that the factor (1/r) is absent, the first term is multiplied by $1/\nu_o^2$, and the second term has a factor $1/\nu^2$ under the integral sign.

CHAPTER VI

EXAMPLES

The Spherical Harmonics Method has been used extensively for the calculation of critical numbers. The examples below will give some indication of the type of convergence that the method gives, and how the critical radius a_1 of a two-medium spherical system varies with the multiplication number f_1 of the core. The mean free paths of the two media are assumed to be equal and f_2 has been taken to be -.05. Table I has also been designed to show how the end-point results differ from the ψ_n-results, and how the exact and the end-point radii compare in the case when f_1 = .5. The exact end-point results were obtained using formula (5.3.) and the graph for Δa_1 in Frankel and Goldberg 1, p. 66.

Table I. Critical Radii.

Approximation	f_1=.25	f_1=.50	end-point f_1=.50 method	f_1=.75	f_1=1.0
ψ_1	2.2994	1.5327	1.5327	1.2157	1.0339
ψ_3	2.1088	1.3426	1.3422	1.0342	.8617
ψ_5	-	1.3147	1.3143	-	-
ψ_7	-	1.3088	1.3084	-	-
end-point method, exact	2.091	1.305	1.305	.981	.795

Table II. Neutron Density.

ψ_1-approximation		ψ_3-approximation		ψ_3-approximation, end-point method	
r	$\psi_0(r)$	r	$\psi_0(r)$	r	$\psi_0(r)$
0.0	1.0000	0.0	1.0000	0.0	1.0000
.2	.9900	.2	.9859	.2	.9859
.4	.9605	.4	.9442	.4	.9442
.6	.9124	.6	.8768	.6	.8768
.8	.8475	.8	.7867	.8	.7867
1.0	.7681	1.0	.6779	1.0	.6778
1.2	.6769	1.2	.5551	1.2	.5548
1.4	.5772	1.3	.4899	1.3	.4896
1.5	.5251	1.3426	.4616	1.3422	.4609
1.5327	.5079	1.4	.4321	1.4	.4312
1.6	.4741	1.5	.3868	1.5	.3861
1.7	.4292	1.7	.3146	1.7	.3139
1.9	.3554	1.9	.2598	1.9	.2593
2.1	.2976	2.1	.2172	2.1	.2168
2.3	.2515	2.3	.1834	2.3	.1831
2.5	.2141	2.5	.1562	2.5	.1559

Table of the $T_n(x,y)$-Matrices.

From paragraph (3.2) we obtain readily the following formulas for the T_n-matrices:

$$T_n(x) = T_n(x,\infty)G_n^{-1},$$

$$T_n^{-1}(x) = G_nT_n(\infty,x),$$

$$T_n(x,y) \equiv T_n(x)T_n(y) = T_n(x,\infty)T_n(\infty,y),$$

$$T_n(x,y)T_n(y,x) = I \quad \text{(the unit matrix)},$$

and

$$T_2(x,y) = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix}, \text{ where } c = (1/y)-(1/x),$$

$$T_3(x,y) = \begin{pmatrix} 1+(4/5)(c/y) & -(4/5)c & 0 & 0 \\ -c+(4/5)(c/xy) & 1-(4/5)(c/x) & & 0 \\ 3(c/y) & -3c & & 1 \end{pmatrix},$$

$$T_4(x,y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/x & 1 & 0 & 0 \\ 0 & 3/x & 1 & 0 \\ 0 & 3/x^2 & 6/x & 1 \end{pmatrix} \cdot$$

$$\cdot\begin{pmatrix} 1 & -(11/7)c & -(99/49)(c/x) & 0 \\ 0 & 1 & -(9/7)c & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/y & 1 & 0 & 0 \\ 3/y^2 & -3/y & 1 & 0 \\ -15/y^3 & 15/y^2 & -6/y & 1 \end{pmatrix},$$

$$T_5(x,y) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/x & 1 & 0 & 0 & 0 \\ 0 & 3/x & 1 & 0 & 0 \\ 0 & 3/x^2 & 6/x & 1 & 0 \\ 0 & 0 & 15/x^2 & 10/x & 1 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} 1 & -2.3333333c & \begin{cases} -4.9470899(c/x) \\ +1.0158730(c/y) \end{cases} & \begin{cases} -\ .7054674c \\ -8.7948226(c/x^2) \end{cases} & 0 \\ 0 & 1 & -2.5555556c & -4.5432099(c/x) & 0 \\ 0 & 0 & 1 & -1.7777778c & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1/y & 1 & 0 & 0 & 0 \\ 3/y^2 & -3/y & 1 & 0 & 0 \\ -15/y^3 & 15/y^2 & -6/y & 1 & 0 \\ 105/y^4 & -105/y^3 & 45/y^2 & -10/y & 1 \end{pmatrix},$$

$$T_6(x,y) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/x & 1 & 0 & 0 & 0 & 0 \\ 0 & 3/x & 1 & 0 & 0 & 0 \\ 0 & 3/x^2 & 6/x & 1 & 0 & 0 \\ 0 & 0 & 15/x^2 & 10/x & 1 & 0 \\ 0 & 0 & 15/x^3 & 45/x^2 & 15/x & 1 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} 1 & -3.0909091c & \begin{cases} +3.0000000c/y \\ -8.8016529c/x \end{cases} & \begin{cases} -1.6528926c \\ -24.1810669c/x^2 \\ +7.0247934c/xy \end{cases} & \begin{cases} -6.3110443c/x \\ -54.9569701c/x^3 \end{cases} & 0 \\ 0 & 1 & -3.8181818c & \begin{cases} -11.2644628c/x \\ +2.2727273c/y \end{cases} & \begin{cases} -.8264463c \\ -25.6010518c/x^2 \end{cases} & 0 \\ 0 & 0 & 1 & -3.5454545c & -8.0578512c/x & 0 \\ 0 & 0 & 0 & 1 & -2.2727273c & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot$$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-1/y & 1 & 0 & 0 & 0 & 0 \\
3/y^2 & -3/y & 1 & 0 & 0 & 0 \\
-15/y^3 & 15/y^2 & -6/y & 1 & 0 & 0 \\
105/y^4 & -105/y^3 & 45/y^2 & -10/y & 1 & 0 \\
-945/y^5 & 945/y^4 & -420/y^3 & 105/y^2 & -15/y & 1
\end{pmatrix} .
$$

Table of the Singular Zeroes of $G_n(f, \mu)$.

f	$\mu_1^{(2)}$	$\mu_1^{(3)}$	$\mu_1^{(4)}$	$\mu_1^{(5)}$	$\mu_1^{(6)}$	$\mu_1^{(\infty)}$
-1.0	.57735	.77459	.86115	.90617	.93247	1.00000
-.9	.60858	.79815	.87783	.91837	.94169	1.00000
-.8	.64550	.82664	.89880	.93428	.95419	1.00009
-.7	.69006	.86189	.92582	.95568	.97173	1.00260
-.6	.74536	.90677	.96172	.98582	.99741	1.01459
-.5	.81650	.96609	1.01158	1.02887	1.03662	1.04439
-.4	.91287	1.04881	1.08385	1.09527	1.09942	1.10215
-.3	1.05410	1.17379	1.19831	1.20447	1.20616	1.20681
-.2	1.29099	1.39045	1.40473	1.40712	1.40754	1.40764
-.1	1.82575	1.89735	1.90273	1.90317	1.90320	1.90320
0	∞	∞	∞	∞	∞	∞
.1	-1.82575i	-1.75119i	-1.75704i	-1.75664i	-1.75667i	-1.75667i
.2	-1.29099i	-1.18321i	-1.20016i	-1.19801i	-1.19829i	-1.19826i
.3	-1.05410i	-.91894i	-.95055i	-.94513i	-.94616i	-.94600i
.4	-.91287i	-.75277i	-.80178i	-.79178i	-.79422i	-.79377i
.5	-.81650i	-.63246i	-.70113i	-.68549i	-.69011i	-.68913i
.6	-.74536i	-.53748i	-.62783i	-.60579i	-.61336i	-.61162i
.7	-.69006i	-.45774i	-.57177i	-.54280i	-.55407i	-.55133i
.8	-.64550i	-.38730i	-.52736i	-.49107i	-.50678i	-.50281i
.9	-.60858i	-.32203i	-.49120i	-.44734i	-.46808i	-.46273i
1.0	-.57735i	-.25820i	-.46112i	-.40949i	-.43585i	-.42898i

f	$1/\mu_1^{(2)}$	$1/\mu_1^{(3)}$	$1/\mu_1^{(4)}$	$1/\mu_1^{(5)}$	$1/\mu_1^{(6)}$	$1/\mu_1^{(\infty)}$
-1.0	1.73205	1.29100	1.16126	1.10354	1.07242	1.00000
-.9	1.64317	1.25290	1.13917	1.08889	1.06192	1.00000
-.8	1.54919	1.20971	1.11259	1.07034	1.04801	.99991
-.7	1.44914	1.16024	1.08012	1.04637	1.02909	.99741
-.6	1.34164	1.10282	1.03980	1.01469	1.00260	.98562
-.5	1.22474	1.03510	.98875	.97194	.96467	.95750
-.4	1.09545	.95346	.92264	.91302	.90957	.90733
-.3	.94868	.85194	.83451	.83024	.82908	.82863
-.2	.77460	.71919	.71188	.71067	.71046	.71041
-.1	.54772	.52705	.52556	.52544	.52543	.52543
0	.00000	.00000	.00000	.00000	.00000	.00000
.1	.54772i	.57104i	.56914i	.56927i	.56926i	.56926i
.2	.77460i	.84516i	.83322i	.83472i	.83452i	.83454i
.3	.94868i	1.08821i	1.05204i	1.05805i	1.05690i	1.05708i
.4	1.09545i	1.32842i	1.24722i	1.26297i	1.25909i	1.25981i
.5	1.22474i	1.58114i	1.42627i	1.45881i	1.44905i	1.45110i
.6	1.34164i	1.86052i	1.59279i	1.65074i	1.63037i	1.63500i
.7	1.44914i	2.18466i	1.74894i	1.84231i	1.80481i	1.81378i
.8	1.54919i	2.58199i	1.89623i	2.03635i	1.97351i	1.98883i
.9	1.64317i	3.10530i	2.03582i	2.23542i	2.13638i	2.16107i
1.0	1.73205i	3.87298i	2.16863i	2.44204i	2.29436i	2.33112i

f	$\sqrt{-3f}$	-3f	$-3f\mu_1$ (2)	$\sqrt{-3f\mu_1}$ (3)	$-3f\mu_1$ (4)	$-3f\mu_1$ (5)	$-3f\mu_1$ (6)	$\sqrt{-3f\mu_1}$
-1.00	1.73205	3.00000	1.00000	1.34163	1.49153	1.56954	1.61509	1.73205
-.99	1.72337	2.97000	1.00000	1.33865	1.48666	1.56356	1.60839	1.72337
-.98	1.71464	2.94000	1.00000	1.33566	1.48178	1.55755	1.60166	1.71464
-.97	1.70587	2.91000	1.00000	1.33267	1.47689	1.55152	1.59493	1.70587
-.96	1.69706	2.88000	1.00000	1.32966	1.47201	1.54549	1.58819	1.69706
-.95	1.68819	2.85000	1.00000	1.32664	1.46710	1.53943	1.58141	1.68819
-.94	1.67929	2.82000	1.00000	1.32363	1.46218	1.53339	1.57464	1.67929
-.93	1.67033	2.79000	1.00000	1.32061	1.45725	1.52731	1.56784	1.67033
-.92	1.66132	2.76000	1.00000	1.31757	1.45232	1.52123	1.56102	1.66132
-.91	1.65227	2.73000	1.00000	1.31454	1.44738	1.51514	1.55420	1.65227
-.90	1.64317	2.70000	1.00000	1.31149	1.44243	1.50903	1.54736	1.64317
-.89	1.63401	2.67000	1.00000	1.30843	1.43747	1.50291	1.54050	1.63401
-.88	1.62481	2.64000	1.00000	1.30537	1.43250	1.49679	1.53364	1.62481
-.87	1.61555	2.61000	1.00000	1.30231	1.42750	1.49065	1.52675	1.61555
-.86	1.60624	2.58000	1.00000	1.29923	1.42252	1.48450	1.51985	1.60624
-.85	1.59687	2.55000	1.00000	1.29615	1.41752	1.47832	1.51295	1.59687
-.84	1.58745	2.52000	1.00000	1.29306	1.41252	1.47216	1.50602	1.58747
-.83	1.57797	2.49000	1.00000	1.28996	1.40751	1.46597	1.49909	1.57800
-.82	1.56844	2.46000	1.00000	1.28686	1.40250	1.45979	1.49215	1.56849
-.81	1.55885	2.43000	1.00000	1.28375	1.39746	1.45360	1.48518	1.55893
-.80	1.54919	2.40000	1.00000	1.28063	1.39242	1.44738	1.47822	1.54933
-.79	1.53948	2.37000	1.00000	1.27749	1.3873[illegible]	1.44116	1.47126	1.53971
-.78	1.52971	2.34000	1.00000	1.27437	1.38233	1.43495	1.46427	1.53006
-.77	1.51987	2.31000	1.00000	1.27122	1.37727	1.42872	1.45728	1.52039
-.76	1.50997	2.28000	1.00000	1.26807	1.37220	1.42247	1.45028	1.51071
-.75	1.50000	2.25000	1.00000	1.26492	1.36712	1.41622	1.44327	1.50102
-.74	1.48997	2.22000	1.00000	1.26175	1.36205	1.40997	1.43625	1.49136
-.73	1.47986	2.19000	1.00000	1.25857	1.35695	1.40371	1.42924	1.48168
-.72	1.46969	2.16000	1.00000	1.25538	1.35185	1.39746	1.42222	1.47205
-.71	1.45945	2.13000	1.00000	1.25220	1.34675	1.39118	1.41520	1.46245
-.70	1.44914	2.10000	1.00000	1.24900	1.34165	1.38492	1.40818	1.45290
-.69	1.43875	2.07000	1.00000	1.24579	1.33652	1.37865	1.40116	1.44340
-.68	1.42829	2.04000	1.00000	1.24259	1.33142	1.37238	1.39415	1.43395
-.67	1.41774	2.01000	1.00000	1.23935	1.32628	1.36609	1.38711	1.42456
-.66	1.40712	1.98000	1.00000	1.23612	1.32115	1.35983	1.38010	1.41526
-.65	1.39642	1.95000	1.00000	1.23288	1.31601	1.35355	1.37308	1.40602
-.64	1.38564	1.92000	1.00000	1.22964	1.31088	1.34728	1.36606	1.39688
-.63	1.37477	1.89000	1.00000	1.22638	1.30574	1.34100	1.3[illegible]909	1.38782
-.62	1.36382	1.86000	1.00000	1.22311	1.30060	1.33474	1.35210	1.37886
-.61	1.35277	1.83000	1.00000	1.21983	1.29543	1.32847	1.34512	1.36999
-.60	1.34164	1.80000	1.00000	1.21655	1.29029	1.32222	1.33816	1.36121
-.59	1.33041	1.77000	1.00000	1.21326	1.28512	1.31596	1.33121	1.35254
-.58	1.31909	1.74000	1.00000	1.20996	1.27997	1.30971	1.32427	1.34395
-.57	1.30767	1.71000	1.00000	1.20665	1.27482	1.30347	1.31735	1.33547
-.56	1.29615	1.68000	1.00000	1.20333	1.26965	1.29724	1.31046	1.32711
-.55	1.28452	1.65000	1.00000	1.20000	1.26449	1.29101	1.30358	1.31884

TABL I

f	$\sqrt{-3f}$	$-3f$	$\sqrt{-3f\mu_1^{(2)}}$	$\sqrt{-3f\mu_1^{(3)}}$	$\sqrt{-3f\mu_1^{(4)}}$	$\sqrt{-3f\mu_1^{(5)}}$	$\sqrt{-3f\mu_1^{(6)}}$	$\sqrt{-3f\mu}$
-.55	1.28452	1.65000	1.00000	1.20000	1.26449	1.29101	1.30358	1.31884
-.54	1.27279	1.62000	1.00000	1.19666	1.25933	1.28480	1.29673	1.31067
-.53	1.26095	1.59000	1.00000	1.19331	1.25415	1.27861	1.28991	1.30262
-.52	1.24900	1.56000	1.00000	1.18997	1.24900	1.27243	1.28310	1.29468
-.51	1.23693	1.53000	1.00000	1.18658	1.24383	1.26626	1.27633	1.28683
-.50	1.22474	1.50000	1.00000	1.18321	1.23868	1.26010	1.26959	1.27910
-.49	1.21244	1.47000	1.00000	1.17984	1.23352	1.25398	1.26289	1.27147
-.48	1.20000	1.44000	1.00000	1.17644	1.22836	1.24787	1.25623	1.26394
-.47	1.18743	1.41000	1.00000	1.17303	1.22321	1.24177	1.24958	1.25651
-.46	1.17473	1.38000	1.00000	1.16961	1.21806	1.23569	1.24300	1.24918
-.45	1.16189	1.35000	1.00000	1.16618	1.21292	1.22964	1.23643	1.24196
-.44	1.14891	1.32000	1.00000	1.16276	1.20[illegible]	1.22361	1.22993	1.23484
-.43	1.13578	1.29000	1.00000	1.15931	1.20265	1.21762	1.22347	1.22782
-.42	1.12250	1.26000	1.00000	1.15586	1.19752	1.21165	1.21705	1.22089
-.41	1.10905	1.23000	1.00000	1.15239	1.19240	1.20570	1.21067	1.21405
-.40	1.09545	1.20000	1.00000	1.14892	1.18730	1.19981	1.20436	1.20733
-.39	1.08167	1.17000	1.00000	1.14542	1.18220	1.19392	1.19810	1.20068
-.38	1.06771	1.14000	1.00000	1.14194	1.17711	1.18807	1.19189	1.19413
-.37	1.05357	1.11000	1.00000	1.13843	1.17203	1.18226	1.18573	1.18768
-.36	1.03923	1.08000	1.00000	1.13490	1.16695	1.17649	1.17963	1.18131
-.35	1.02470	1.05000	1.00000	1.13138	1.16190	1.17075	1.17358	1.17503
-.34	1.00995	1.02000	1.00000	1.12783	1.15685	1.16505	1.16759	1.16883
-.33	.99499	.99000	1.00000	1.12428	1.15182	1.15939	1.16167	1.16272
-.32	.97980	.96000	1.00000	1.12072	1.14681	1.15378	1.15581	1.15669
-.31	.96437	.93000	1.00000	1.11715	1.14181	1.14820	1.15002	1.15074
-.30	.94868	.90000	1.00000	1.11355	1.13681	1.14266	1.14426	1.14488
-.29	.93274	.87000	1.00000	1.10996	1.13186	1.13717	1.13857	1.13910
-.28	.91652	.84000	1.00000	1.10636	1.12692	1.13174	1.13296	1.13338
-.27	.90000	.81000	1.00000	1.10272	1.12197	1.12634	1.12740	1.12775
-.26	.88318	.78000	1.00000	1.09910	1.11707	1.1[illegible]100	1.12191	1.12220
-.25	.86603	.75000	1.00000	1.09545	1.11219	1.11570	1.11648	1.11671
-.24	.84853	.72000	1.00000	1.09179	1.10732	1.11045	1.11111	1.11130
-.23	.83066	.69000	1.00000	1.08812	1.10248	1.10525	1.10581	1.10595
-.22	.81240	.66000	1.00000	1.08443	1.09766	1.10010	1.10056	1.10068
-.21	.79373	.63000	1.00000	1.08076	1.09287	1.09501	1.09540	1.09550
-.20	.77460	.60000	1.00000	1.07705	1.08810	1.08996	1.09028	1.09036
-.19	.75498	.57000	1.00000	1.07331	1.08336	1.08496	1.08522	1.08527
-.18	.73485	.54000	1.00000	1.06959	1.07866	1.08003	1.08025	1.08028
-.17	.71414	.51000	1.00000	1.06583	1.07399	1.07514	1.07530	1.07533
-.16	.69282	.48000	1.00000	1.06207	1.06933	1.07031	1.07044	1.07046
-.15	.67082	.45000	1.00000	1.05830	1.06473	1.06552	1.06562	1.06564
-.14	.64807	.42000	1.00000	1.05451	1.06013	1.06079	1.06086	1.06088
-.13	.62449	.39000	1.00000	1.05069	1.05558	1.05611	1.05615	1.05617
-.12	.60000	.36000	1.00000	1.04690	1.05108	1.05151	1.05152	1.05154
-.11	.57446	.33000	1.00000	1.04307	1.04662	1.04693	1.04695	1.04695
-.10	.54772	.30000	1.00000	1.03923	1.04217	1.04241	1.04243	1.04243

x	$\sqrt{-3x}$	$-3x$	[illegible] μ_1	[illegible] μ_1	[illegible] μ_1	[illegible] μ_1	[illegible] μ_1	[illegible]
-.10	.54772	.30000	1.00000	1.03923	1.04217	1.04241	1.04243	1.04243
-.09	.51962	.27000	1.00000	1.03537	1.03777	1.03794	1.03796	1.03796
-.08	.48990	.24000	1.00000	1.03150	1.03341	1.03353	1.03354	1.03354
-.07	.45826	.21000	1.00000	1.02761	1.02908	1.02917	1.02917	1.02917
-.06	.42426	.18000	1.00000	1.02371	1.02480	1.02485	1.02485	1.02485
-.05	.38730	.15000	1.00000	1.01980	1.02056	1.02059	1.02059	1.02059
-.04	.34641	.12000	1.00000	1.01587	1.01636	1.01638	1.01638	1.01638
-.03	.30000	.09000	1.00000	1.01193	1.01220	1.01221	1.01221	1.01221
-.02	.24495	.06000	1.00000	1.00797	1.00809	1.00809	1.00809	1.00809
-.01	.17321	.03000	1.00000	1.00399	1.00402	1.00402	1.00402	1.00402
.00	.00000	.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
.01	.17321i	-.03000	1.00000	.99599	.99602	.99602	.99602	.99602
.02	.24495i	-.06000	1.00000	.99197	.99209	.99209	.99209	.99209
.03	.30000i	-.09000	1.00000	.98793	.98821	.98820	.98820	.98820
.04	.34641i	-.12000	1.00000	.98387	.98437	.98435	.98436	.98436
.05	.38730i	-.15000	1.00000	.97980	.98058	.98055	.98056	.98055
.06	.42426i	[illegible]	1.00000	.97571	.97684	.97679	.97680	.97680
.07	.45826i	-.21000	1.00000	.97160	.97315	.97307	.97308	.97308
.08	.48990i	-.24000	1.00000	.96748	.96951	.96939	.96940	.96940
.09	.51962i	-.27000	1.00000	.96333	.96591	.96574	.96576	.96576
.10	.54772i	-.30000	1.00000	.95917	.96237	.96214	.96216	.96216
.11	.56446i	-.33000	1.00000	.95500	.95887	.95858	.95860	.95860
.12	.60000i	-.36000	1.00000	.95080	.95543	.95505	.95508	.95508
.13	.62450i	-.39000	1.00000	.94657	.95204	.95155	.95159	.95159
.14	.64807i	-.42000	1.00000	.94233	.94868	.94808	.94814	.94814
.15	.67082i	-.45000	1.00000	.93808	.94539	.94465	.94473	.94472
.16	.69282i	-.48000	1.00000	.93381	.94214	.94[illegible]25	.94136	.94134
.17	.71414i	-.51000	1.00000	.92951	.93893	.93789	.93802	.93800
.18	.73485i	-.54000	1.00000	.92521	.935[illegible]	.934[illegible]	.93471	.93470
.19	.75498i	-.57000	1.00000	.92086	.93268	.93125	.93143	.93141
.20	.77460i	-.60000	1.00000	.91651	.92965	.92798	.92820	.92[illegible]18
.21	.79373i	-.63000	1.00000	.91214	.92665	.92473	.92500	.92496
.22	.81240i	-.66000	1.00000	.90774	.92369	.92150	.92182	.92178
.23	.83066i	-.69000	1.00000	.90332	.92078	.91830	.91869	.91864
.24	.84853i	-.72000	1.00000	.89889	.91793	.91514	.91558	.91553
.25	.86603i	-.75000	1.00000	.89443	.91512	.91200	.91251	.91245
.26	.88318i	-.78000	1.00000	.88994	.91236	.90888	.90947	.90939
.27	.90000i	-.81000	1.00000	.88544	.90963	.90579	.90645	.90635
.28	.91652i	-.84000	1.00000	.88091	.90697	.90272	.90348	.90337
.29	.93274i	-.87000	1.00000	.87636	.90434	.89966	.90053	.90040
.30	.94868i	-.90000	1.00000	.87178	.90175	.89663	.89761	.89745
.31	.96437i	-.93000	1.00000	.86718	.89923	.89362	.89472	.89455
.32	.97980i	-.96000	1.00000	.86256	.89674	.89064	.89186	.89167
.33	.99499i	-.99000	1.00000	.85791	.89428	.88766	.88903	.88880
.34	1.00995i	-1.02000	1.00000	.85323	.89187	.88471	.88623	.88598
.35	1.02470i	-1.05000	1.00000	.84854	.88952	.88178	.88346	.88317
.36	1.03923i	-1.08000	1.00000	.84380	.88719	.87886	.88072	.88039

TABLE I

f	$\sqrt{-3f}$	$-3f$	$\sqrt{-3f\mu_1}^{(2)}$	$\sqrt{-3f\mu_1}^{(3)}$	$\sqrt{-3f\mu_1}^{(4)}$	$\sqrt{-3f\mu_1}^{(5)}$	$\sqrt{-3f\mu_1}^{(6)}$	$\sqrt{-3f\mu}$
.36	1.03923i	-1.08000	1.00000	.84380	.88719	.87886	.88072	.88039
.37	1.05357i	-1.11000	1.00000	.83905	.88491	.87596	.87801	.87764
.38	1.06771i	-1.14000	1.00000	.83427	.88267	.87308	.87532	.87491
.39	1.08167i	-1.17000	1.00000	.82946	.88047	.87021	.87267	.87222
.40	1.09545i	-1.20000	1.00000	.82463	.87831	.86736	.87003	.86954
.41	1.10905i	-1.23000	1.00000	.81975	.87619	.86451	.86743	.86688
.42	1.12250i	-1.26000	1.00000	.81486	.87411	.86169	.86485	.86425
.43	1.13578i	-1.29000	1.00000	.80994	.87205	.85888	.86230	.86163
.44	1.14891i	-1.32000	1.00000	.80498	.87004	.85609	.85978	.85905
.45	1.16189i	-1.35000	1.00000	.80000	.86807	.85330	.85728	.85648
.46	1.17473i	-1.38000	1.00000	.79498	.86612	.85053	.85482	.85395
.47	1.18743i	-1.41000	1.00000	.78993	.86421	.84776	.85237	.85143
.48	1.20000i	-1.44000	1.00000	.78486	.86235	.84502	.84996	.84894
.49	1.21244i	-1.47000	1.00000	.77975	.86052	.84228	.84757	.84646
.50	1.22474i	-1.50000	1.00000	.77459	.[illegible]870	.83955	.84520	.84401
.51	1.23693i	-1.53000	1.00000	.76941	.85693	.83683	.84286	.84157
.52	1.24900i	-1.56000	1.00000	.76420	.85519	.83412	.84055	.83917
.53	1.20695i	-1.59000	1.00000	.75895	.85348	.83143	.83826	.83677
.54	1.27279i	-1.62000	1.00000	.75366	.85180	.82873	.83599	.83440
.55	1.28452i	-1.65000	1.00000	.74833	.85015	.82605	.83375	.83205
.56	1.29615i	-1.68000	1.00000	.74297	.84853	.82338	.83154	.82972
.57	1.30767i	-1.71000	1.00000	.73756	.84694	.82071	.82934	.82740
.58	1.31909i	-1.74000	1.00000	.73212	.84537	.81805	.82717	.82511
.59	1.33041i	-1.77000	1.00000	.72663	.84383	.81540	.82502	.82283
.60	1.34164i	-1.80000	1.00000	.72111	.84232	.81275	.82291	.82057
.61	1.35277i	-1.83000	1.00000	.71554	.84084	.81011	.82081	.81833
.62	1.36382i	-1.86000	1.00000	.70993	.83938	.80748	.81873	.81612
.63	1.37477i	-1.89000	1.00000	.70427	.83794	.80485	.81668	.81391
.64	1.38564i	-1.92000	1.00000	.69857	.83654	.80223	.81465	.81172
.65	1.39642i	-1.95000	1.00000	.69282	.83515	.79961	.81264	.80955
.66	1.40712i	-1.98000	1.00000	.68702	.83379	.79700	.81065	.80740
.67	1.41774i	-2.01000	1.00000	.68117	.83245	.79439	.80869	.80526
.68	1.42829i	-2.[illegible]000	1.00000	.67528	.83115	.79179	.80675	.80315
.69	1.43875i	-[illegible]7000	1.00000	.66933	.82985	.78918	.80483	.80105
.70	1.44914i	-2.10000	1.00000	.66333	.82858	.78659	.80293	.79896
.71	1.45945i	-2.13000	1.00000	.65726	.82733	.78399	.80105	.79689
.72	1.46969i	-2.16000	1.00000	.65115	.82610	.78140	.79919	.79484
.73	1.47986i	-2.19000	1.00000	.64498	.82489	.77881	.79736	.79279
.74	1.48997i	-2.22000	1.00000	.63875	.82371	.77623	.79555	.79077
.75	1.50000i	-2.25000	1.00000	.63246	.82254	.77365	.79375	.78877
.76	1.50997i	-2.28000	1.00000	.62610	.8[illegible]139	.77107	.79197	.78677
.77	1.51987i	-2.31000	1.00000	.61968	.8[illegible]27	.76849	.79[illegible]	.78479
.78	1.52971i	-2.34000	1.00000	[illegible]	[illegible]	[illegible]	[illegible]	.78283
.79	1.53948i	-2.37000	1.00000	.60663	.81806	.76334	.78677	.78088
.80	1.54919i	-2.40000	1.00000	.60000	.81698	.76077	.78507	.77895
.81	1.55885i	-2.43000	1.00000	.59330	.81593	.75820	.78340	.77703

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TABL[illegible] I

f	$\sqrt{-3f}$	$-3f$	$\sqrt{-3f\mu_1}^{(2)}$	$\sqrt{-3f\mu_1}^{(3)}$	$\sqrt{-3f\mu_1}^{(4)}$	$\sqrt{-3f\mu_1}^{(5)}$	$\sqrt{-3f\mu_1}^{(6)}$	$\sqrt{-3f\mu}$
.81	1.55885i	-2.43000	1.00000	.59330	.81593	.75820	.78340	.77703
.82	1.56844i	-2.46000	1.00000	.58651	.8148[illegible]	.75562	.78174	.77512
.83	1.57797i	-2.49000	1.00000	.57965	.81386	.75305	.78010	.77323
.84	1.58745i	-2.52000	1.00000	.57271	.81285	.75048	.77848	.77135
.85	1.59687i	-2.55000	1.00000	.56568	.81186	.74791	.77688	.76948
.86	1.60624i	-2.58000	1.00000	.55857	.81089	.74534	.77530	.76760
.87	1.61555i	-2.61000	1.00000	.55136	.80993	.74277	.77373	.76579
.88	1.62481i	-2.64000	1.00000	.54406	.80898	.74021	.77218	.76397
.89	1.63401i	-2.67000	1.00000	.53666	.80804	.73763	.77065	.76215
.90	1.64317i	-2.70000	1.00000	.52915	.80713	.73506	.76914	.76035
.91	1.65227i	-2.73000	1.00000	.52154	.80623	.73249	.76764	.75856
.92	1.66132i	-2.76000	1.00000	.51381	.80533	.72991	.76616	.75679
.93	1.67033i	-2.79000	1.00000	.50596	.80446	.72734	.76470	.75502
.94	1.67929i	-2.82000	1.00000	.49800	.80360	.72477	.76325	.75327
.95	1.68819i	-2.85000	1.00000	.48990	.80275	.72218	.76182	.75153
.96	1.69706i	-2.88000	1.00000	.48167	.80191	.71961	.76041	.74981
.97	1.70587i	-2.91000	1.00000	.47329	.80108	.71702	.75901	.74809
.98	1.71464i	-2.94000	1.00000	.46476	.80027	.71444	.75[illegible]63	.74638
.99	1.72337i	-2.97000	1.00000	.45607	.79947	.71185	.75627	.74469
1.00	1.73205i	-3.00000	1.00000	.44721	.79868	.70926	.75492	.74301
1.1	1.81659i	-3.30000	1.00000	.34641	.79140	.68320	.74222	.72680
1.2	1.89737i	-3.60000	1.00000	.20000	.78506	.65671	.73088	.71159
1.3	1.97484i	-3.90000	1.00000		.77950	.62963	.72072	.69729
1.4	2.04939i	-4.20000	1.00000		.77460	.60178	.71161	.68382
1.5	2.12132i	-4.50000	1.00000		.77025	.57299	.70340	.67109
1.6	2.19089i	-4.80000	1.00000		.76636	.54303	.69600	.65904
1.7	2.25832i	-5.10000	1.00000		.76287	.51164	.68930	.64762
1.8	2.32379i	-5.40000	1.00000		.75973	.47851	.68322	.63677
1.9	2.38747i	-5.70000	1.00000		.75688	.44319	.67768	.62644
2.0	2.44949i	-6.00000	1.00000		.75428	.40508	.67261	.61660
2.1	2.50998i	-6.30000	1.00000		.75191	.36324	.66797	.60720
2.2	2.56905i	-6.60000	1.00000		.74974	.31619	.66371	.59823
2.3	2.62679i	-6.90000	1.00000		.74774	.26108	.65979	.58963
2.4	2.68328i	-7.20000	1.00000		.74589	.19100	.65616	.58142
2.5	2.73861i	-7.50000	1.00000		.74418	.07015	.65280	.57350
3.0	3.00000i	-9.00000	1.00000		.73725		.63918	.53831
4.0	3.46410i	-12.0000	1.00000		.72838		.62186	.48380
5.0	3.87298i	-15.0000	1.00000		.72295		.61138	.44306
6.0	4.24264i	-1[illegible].000[illegible]	1.00000		.71930		.60438	.41113
7.0	4.58258i	-21.0000	1.00000		.71667		.59939	.38524
8.0	4.89898i	-24.0000	1.00000		.71469		.59566	.36369
9.0	5.19615i	-27.0000	1.00000		.71315		.59275	.34539
10.0	5.47723i	-30.0000	1.00000		.71191		.59044	.32960
∞	∞i	-∞	1.00000		.70065		.56900	.[illegible]0000

f	$\mu_2^{(4)}$	$\mu_2^{(5)}$	$\mu_2^{(6)}$	$\mu_3^{(6)}$
-1.00	.33998	.53847	.66121	.23862
- .99	.34109	.53976	.66241	.23918
- .98	.34222	.54107	.66363	.23974
- .97	.34335	.54240	.66486	.24031
- .96	.34449	.54373	.66610	.24088
- .95	.34564	.54508	.66736	.24145
- .94	.34681	.54645	.66865	.24203
- .93	.34798	.54782	.66994	.24261
- .92	.34916	.54922	.67125	.24319
- .91	.35035	.55062	.67258	.24377
- .90	.35156	.55204	.67392	.24436
- .89	.35277	.55348	.67528	.24496
- .88	.35399	.55493	.67666	.24555
- .87	.35523	.55639	.67806	.24615
- .86	.35647	.55787	.67947	.24676
- .85	.35773	.55937	.68090	.24736
- .84	.35900	.56087	.68235	.24797
- .83	.36028	.56240	.68381	.24859
- .82	.36157	.56394	.68529	.24920
- .81	.36287	.56550	.68679	.24982
- .80	.36418	.56707	.68831	.25045
- .79	.36551	.56865	.68985	.25107
- .78	.36684	.57026	.69140	.25171
- .77	.36819	.57187	.69297	.25234
- .76	.36955	.57351	.69456	.25298
- .75	.37092	.57516	.69616	.25362
- .74	.37230	.57682	.69778	.25427
- .73	.37370	.57850	.69942	.25492
- .72	.37511	.58020	.70108	.25557
- .71	.37653	.58191	.70275	.25622
- .70	.37796	.58364	.70444	.25689
- .69	.37941	.58538	.70614	.25755
- .68	.38087	.58714	.70786	.25822
- .67	.38234	.58891	.70960	.25889
- .66	.38383	.59070	.71135	.25956
- .65	.38532	.59250	.71312	.26024
- .64	.38683	.59432	.71490	.26093
- .63	.38836	.59615	.71670	.26161
- .62	.38989	.59800	.71850	.26230
- .61	.39144	.59986	.72033	.26300
- .60	.39301	.60174	.72216	.26369
- .59	.39458	.60362	.72400	.26440
- .58	.39617	.60553	.72585	.26510
- .57	.39778	.60744	.72772	.26581
- .56	.39939	.60937	.72959	.26652
- .55	.40103	.61131	.73147	.26724

ρ	$\mu_2^{(4)}$	$\mu_2^{(5)}$	$\mu_2^{(6)}$	$\mu_3^{(6)}$
-.55	.40103	.61131	.73147	.26724
-.54	.40267	.61326	.73336	.26796
-.53	.40433	.61523	.73525	.26869
-.52	.40600	.61720	.73715	.26942
-.51	.40768	.61918	.73905	.27015
-.50	.40938	.62118	.74095	.27089
-.49	.41109	.62318	.74285	.27162
-.48	.41282	.62519	.74476	.27237
-.47	.41456	.62721	.74666	.27312
-.46	.41631	.62924	.74856	.27387
-.45	.41807	.63127	.75046	.27462
-.44	.41985	.63330	.75235	.27538
-.43	.42165	.63535	.75423	.27615
-.42	.42345	.63739	.75611	.27691
-.41	.42527	.63944	.75798	.27768
-.40	.42710	.64149	.75984	.27846
-.39	.42894	.64355	.76168	.27924
-.38	.43080	.64560	.76351	.28002
-.37	.43266	.64765	.76533	.28081
-.36	.43454	.64970	.76713	.28159
-.35	.43644	.65175	.76892	.28239
-.34	.43834	.65380	.77068	.28319
-.33	.44025	.65584	.77243	.28399
-.32	.44218	.65787	.77416	.28479
-.31	.44412	.65990	.77587	.28560
-.30	.44606	.66192	.77755	.28641
-.29	.44802	.66393	.77922	.28723
-.28	.44999	.66593	.78085	.28804
-.27	.45197	.66792	.78247	.28887
-.26	.45395	.66990	.78406	.28969
-.25	.45594	.67187	.78563	.29052
-.24	.45795	.67382	.78716	.29135
-.23	.45996	.67577	.78867	.29219
-.22	.46198	.67769	.79016	.29303
-.21	.46400	.67960	.79162	.29387
-.20	.46603	.68149	.79305	.29471
-.19	.46807	.68336	.79445	.29556
-.18	.47011	.68522	.79583	.29641
-.17	.47216	.68706	.79718	.29727
-.16	.47421	.68888	.79851	.29813
-.15	.47627	.69067	.79980	.29899
-.14	.47832	.69245	.80107	.29985
-.13	.48038	.69420	.80231	.30072
-.12	.48245	.69594	.80353	.30159
-.11	.48451	.69765	.80472	.30246
-.10	.48657	.69933	.80588	.30333

x	$\mu_2^{(4)}$	$\mu_2^{(5)}$	$\mu_2^{(6)}$	$\mu_3^{(6)}$
- .10	.48657	.69933	.80588	.30333
- .09	.48864	.70100	.80702	.30421
- .08	.49070	.70264	.80813	.30509
- .07	.49276	.70426	.80923	.30597
- .06	.49482	.70586	.81029	.30686
- .05	.49688	.70743	.81133	.30775
- .04	.49893	.70897	.81235	.30863
- .03	.50098	.71049	.81334	.30952
- .02	.50302	.71200	.81431	.31042
- .01	.50506	.71347	.81526	.31132
.00	.50709	.71492	.81619	.31221
.01	.50912	.71635	.81709	.31311
.02	.51114	.71775	.81798	.31401
.03	.51314	.71913	.81884	.31492
.04	.51514	.72048	.81969	.31582
.05	.51713	.72182	.82051	.31672
.06	.51911	.72312	.82132	.31763
.07	.52108	.72441	.82211	.31854
.08	.52304	.72567	.82288	.31945
.09	.52499	.72691	.82364	.32036
.10	.52692	.72813	.82437	.32127
.11	.52884	.72933	.82509	.32218
.12	.53075	.73050	.82580	.32309
.13	.53264	.73165	.82648	.32401
.14	.53452	.73278	.82716	.32492
.15	.53639	.73390	.82781	.32584
.16	.53824	.73499	.82846	.32675
.17	.54007	.73606	.82908	.32767
.18	.54189	.73711	.82970	.32858
.19	.54369	.73814	.83030	.32950
.20	.54547	.73915	.83090	.33041
.21	.54724	.74014	.83147	.33132
.22	.54899	.74112	.83204	.33224
.23	.55072	.74207	.83259	.33315
.24	.55243	.74301	.83313	.33407
.25	.55413	.74393	.83366	.33498
.26	.55581	.74483	.83417	.33589
.27	.55747	.74572	.83468	.33680
.28	.55911	.74660	.83518	.33771
.29	.56073	.74745	.83567	.33862
.30	.56233	.74829	.83614	.33953
.31	.56392	.74911	.83661	.34043
.32	.56549	.74992	.83707	.34134
.33	.56703	.75072	.83751	.34224
.34	.56857	.75150	.83796	.34314
.35	.57008	.75226	.83838	.34404
.36	.57157	.75301	.83881	.34494

TABLE II

r	$\mu_2^{(4)}$	$\mu_2^{(5)}$	$\mu_2^{(6)}$	$\mu_3^{(6)}$
.36	.57157	.75301	.83881	.34494
.37	.57304	.75375	.83922	.34583
.38	.57450	.75448	.83963	.34673
.39	.57593	.75518	.84003	.34762
.40	.57735	.75589	.84042	.34850
.41	.57875	.75657	.84080	.34939
.42	.58013	.75725	.84118	.35027
.43	.58149	.75791	.84155	.35115
.44	.58284	.75856	.84191	.35203
.45	.58416	.75920	.84227	.35291
.46	.58547	.75983	.84262	.35378
.47	.58676	.76045	.84296	.35465
.48	.58803	.76106	.84330	.35552
.49	.58929	.76165	.84363	.35638
.50	.59053	.76224	.84395	.35724
.51	.59175	.76282	.84427	.35810
.52	.59296	.76339	.84459	.35895
.53	.59415	.76394	.84490	.35980
.54	.59532	.76449	.84520	.36064
.55	.59647	.76503	.84550	.36149
.56	.59761	.76556	.84579	.36232
.57	.59874	.76608	.84608	.36316
.58	.59985	.76659	.84636	.36399
.59	.60094	.76710	.84664	.36482
.60	.60202	.76760	.84691	.36564
.61	.60308	.76808	.84718	.36646
.62	.60413	.76856	.84745	.36727
.63	.60516	.76904	.84771	.36808
.64	.60618	.76950	.84796	.36889
.65	.60718	.76997	.84822	.36969
.66	.60818	.77042	.84846	.37049
.67	.60915	.77086	.84871	.37128
.68	.61011	.77130	.84895	.37207
.69	.61106	.77172	.84918	.37285
.70	.61200	.77215	.84942	.37363
.71	.61293	.77256	.84965	.37440
.72	.61384	.77298	.84987	.37517
.73	.61473	.77338	.85009	.37594
.74	.61562	.77378	.85031	.37670
.75	.61649	.77417	.85053	.37746
.76	.61736	.77456	.85075	.37821
.77	.61821	.77495	.85095	.37895
.78	.61905	.77532	.85116	.37970
.79	.61987	.77569	.85136	.38043
.80	.62069	.77605	.85156	.38116
.81	.62149	.77642	.85176	.38189

TABLE II

f	$\mu_2^{(4)}$	$\mu_2^{(5)}$	$\mu_2^{(6)}$	$\mu_3^{(6)}$
.81	.62149	.77642	.85176	.38189
.82	.62229	.77677	.85196	.38261
.83	.62307	.77712	.85215	.38333
.84	.62384	.77747	.85233	.38405
.85	.62461	.77780	.85252	.38475
.86	.62536	.77814	.85270	.38546
.87	.62610	.77847	.85289	.38615
.88	.62683	.77880	.85307	.38685
.89	.62755	.77912	.85324	.38754
.90	.62827	.77944	.85342	.38822
.91	.62897	.77975	.85359	.38890
.92	.62967	.78006	.85376	.38957
.93	.63035	.78037	.85392	.39024
.94	.63103	.78067	.85409	.39090
.95	.63170	.78096	.85425	.39156
.96	.63236	.78126	.85441	.39222
.97	.63300	.78154	.85457	.39286
.98	.63365	.78183	.85473	.39351
.99	.63428	.78211	.85488	.39415
1.00	.63491	.78239	.85503	.39478
1.1	.64076	.78498	.85646	.40087
1.2	.64593	.78727	.85769	.40649
1.3	.65054	.78931	.85884	.41168
1.4	.65466	.79114	.85984	.41647
1.5	.65835	.79279	.86075	.42088
1.6	.66169	.79428	.86157	.42495
1.7	.66471	.79563	.86232	.42871
1.8	.66747	.79687	.86300	.43218
1.9	.66998	.79800	.86363	.43540
2.0	.67229	.79904	.86421	.43839
2.1	.67440	.79999	.86474	.44116
2.2	.67636	.80088	.86523	.44374
2.3	.67817	.80170	.86569	.44614
2.4	.67985	.80247	.86612	.44839
2.5	.68141	.80318	.86652	.45049
3.0	.68781		.86817	.45921
4.0	.69619		.87036	.47081
5.0	.70142		.87176	.47812
6.0	.70498		.87272	.48312
7.0	.70756		.87342	.48675
8.0	.70953		.87396	.48950
9.0	.71106		.87439	.49166
10.0	.71230		.87473	.49339
∞	.72375		.87798	.50937

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f	$-3f[\mu_1^{(3)}]$	$-3f[\mu_1^{(4)}]$	$-3f[\mu_1^{(5)}]$	$-3f[\mu_1^{(6)}]$	$[\mu_2^{(4)}]$	$[\mu_2^{(5)}]$	$[\mu_2^{(6)}]$	$[\mu_3^{(6)}]$
-1.00	1.79997	2.22466	2.46346	2.60852	.11559	.28995	.43720	.05694
- .99	1.79198	2.21016	2.44472	2.58692	.11634	.29134	.43879	.05721
- .98	1.78399	2.19567	2.42596	2.56531	.11711	.29276	.44040	.05748
- .97	1.77601	2.18120	2.40721	2.54380	.11789	.29420	.44204	.05775
- .96	1.76800	2.16681	2.38854	2.52235	.11867	.29564	.44369	.05802
- .95	1.75997	2.15238	2.36984	2.50086	.11947	.29711	.44537	.05830
- .94	1.75200	2.13797	2.35128	2.47949	.12028	.29861	.44709	.05858
- .93	1.74401	2.12358	2.33268	2.45812	.12109	.30011	.44882	.05886
- .92	1.73599	2.10923	2.31414	2.43678	.12191	.30164	.45058	.05914
- .91	1.72801	2.09491	2.29565	2.41554	.12275	.30318	.45236	.05942
- .90	1.72001	2.08060	2.27717	2.39432	.12359	.30475	.45417	.05971
- .89	1.71199	2.06632	2.25874	2.37314	.12445	.30634	.45600	.06001
- .88	1.70399	2.05206	2.24038	2.35205	.12531	.30795	.45787	.06029
- .87	1.69601	2.03776	2.22204	2.33097	.12619	.30957	.45977	.06059
- .86	1.68800	2.02356	2.20374	2.30994	.12707	.31122	.46168	.06089
- .85	1.68000	2.00936	2.18543	2.28902	.12797	.31289	.46362	.06119
- .84	1.67200	1.99521	2.16726	2.26810	.12888	.31458	.46560	.06149
- .83	1.66400	1.98108	2.14907	2.24727	.12980	.31629	.46760	.06180
- .82	1.65601	1.96701	2.13099	2.22651	.13073	.31803	.46962	.06210
- .81	1.64801	1.95289	2.11295	2.20576	.13167	.31979	.47168	.06241
- .80	1.64001	1.93883	2.09491	2.18513	.13263	.32157	.47377	.06273
- .79	1.63198	1.92480	2.07694	2.16461	.13360	.32336	.47589	.06304
- .78	1.62402	1.91084	2.05908	2.14409	.13457	.32520	.47803	.06334
- .77	1.61600	1.89687	2.04124	2.12366	.13556	.32704	.48021	.06368
- .76	1.60800	1.88293	2.02342	2.10331	.13657	.32891	.48241	.06400
- .75	1.60002	1.86902	2.00568	2.08303	.13758	.33081	.48464	.06432
- .74	1.59201	1.85518	1.98802	2.06281	.13861	.33272	.48690	.06465
- .73	1.58400	1.84131	1.97040	2.04273	.13965	.33466	.48919	.06498
- .72	1.57598	1.82750	1.95289	2.02271	.14071	.33663	.49151	.06532
- .71	1.56800	1.81374	1.93538	2.00279	.14177	.33862	.49386	.06565
- .70	1.5600	1.80002	1.91800	1.98297	.14285	.34064	.49624	.06599
- .69	1.55199	1.78629	1.90068	1.96325	.14395	.34267	.49863	.06633
- .68	1.54403	1.77268	1.88343	1.94365	.14506	.34473	.50107	.06668
- .67	1.53599	1.75902	1.86620	1.92407	.14618	.34681	.50353	.06702
- .66	1.52799	1.74544	1.84914	1.90468	.14733	.34893	.50602	.06737
- .65	1.51999	1.73188	1.83210	1.88535	.14847	.35106	.50854	.06772
- .64	1.51201	1.71841	1.81516	1.86617	.14964	.35322	.51108	.06808
- .63	1.50401	1.70496	1.79828	1.84713	.15082	.35539	.51366	.06844
- .62	1.49600	1.69156	1.78153	1.82817	.15201	.35760	.51624	.06880
- .61	1.48799	1.67814	1.76483	1.80935	.15323	.35983	.51888	.06917
- .60	1.47999	1.66485	1.74827	1.79067	.15446	.36209	.52152	.06953
- .59	1.47200	1.65153	1.73175	1.77212	.15569	.36436	.52418	.06991
- .58	1.46400	1.63832	1.71534	1.75369	.15695	.36667	.52686	.07028
- .57	1.45600	1.62517	1.69903	1.73541	.15823	.36898	.52956	.07065
- .56	1.44800	1.61201	1.68283	1.71731	.15951	.37133	.53230	.07103
- .55	1.44000	1.59893	1.66671	1.69932	.16083	.37370	.53505	.07142

z	$-3f[\mu_1^{(3)}]^2$	$-3f[\mu_1^{(4)}]^2$	$-3f[\mu_1^{(5)}]^2$	$-3f[\mu_1^{(6)}]^2$	$[\mu_2^{(4)}]^2$	$[\mu_2^{(5)}]^2$	$[\mu_2^{(6)}]^2$	$[\mu_3^{(6)}]^2$
- .55	1.44000	1.59893	1.66671	1.69932	.16083	.37370	.53505	.07142
- .54	1.43200	1.58591	1.65071	1.68151	.16214	.37609	.53782	.07180
- .53	1.42399	1.57289	1.63484	1.66387	.16348	.37851	.54059	.07219
- .52	1.41603	1.56000	1.61908	1.64635	.16484	.38094	.54339	.07259
- .51	1.40797	1.54711	1.60341	1.62901	.16620	.38338	.54619	.07298
- .50	1.39999	1.53433	1.58785	1.61186	.16759	.38586	.54901	.07338
- .49	1.39202	1.52157	1.57247	1.59489	.16899	.38835	.55183	.07378
- .48	1.38401	1.50887	1.55718	1.57811	.17042	.39086	.55467	.07419
- .47	1.37600	1.49624	1.54199	1.56145	.17186	.39339	.55750	.07459
- .46	1.36799	1.48367	1.52693	1.54505	.17331	.39594	.56034	.07500
- .45	1.35998	1.47117	1.51201	1.52876	.17478	.39850	.56319	.07542
- .44	1.35201	1.45873	1.49722	1.51273	.17627	.40107	.56603	.07583
- .43	1.34400	1.44637	1.48260	1.49688	.17779	.40367	.56886	.07626
- .42	1.33601	1.43405	1.46810	1.48121	.17931	.40627	.57170	.07668
- .41	1.32800	1.42182	1.45371	1.46572	.18085	.40888	.57453	.07711
- .40	1.32002	1.40968	1.43954	1.45048	.18241	.41151	.57736	.07754
- .39	1.31199	1.39760	1.42544	1.43544	.18399	.41416	.58016	.07797
- .38	1.30403	1.38559	1.41151	1.42060	.18559	.41680	.58295	.07841
- .37	1.29602	1.37365	1.39774	1.40596	.18719	.41945	.58573	.07885
- .36	1.28800	1.36177	1.38413	1.39153	.18883	.42211	.58849	.07929
- .35	1.28002	1.35001	1.37066	1.37729	.19048	.42478	.59124	.07974
- .34	1.27200	1.33830	1.35734	1.36327	.19214	.42745	.59395	.08020
- .33	1.26401	1.32669	1.34419	1.34948	.19382	.43013	.59665	.08065
- .32	1.25601	1.31517	1.33121	1.33590	.19552	.43279	.59932	.08111
- .31	1.24802	1.30373	1.31836	1.32255	.19724	.43547	.60197	.08157
- .30	1.23999	1.29234	1.30567	1.30933	.19897	.43814	.60458	.08203
- .29	1.23201	1.28111	1.29316	1.29634	.20072	.44080	.60718	.08250
- .28	1.22403	1.26995	1.28084	1.28360	.20249	.44346	.60973	.08297
- .27	1.21599	1.25882	1.26864	1.27103	.20428	.44612	.61226	.08345
- .26	1.20802	1.24785	1.25664	1.25868	.20607	.44877	.61475	.08392
- .25	1.20001	1.23697	1.24479	1.24653	.20788	.45141	.61721	.08440
- .24	1.19201	1.22616	1.23310	1.23457	.20972	.45403	.61962	.08488
- .23	1.18401	1.21546	1.22158	1.22282	.21156	.45667	.62200	.08537
- .22	1.17599	1.20486	1.21022	1.21123	.21343	.45926	.62435	.08587
- .21	1.16804	1.19436	1.19905	1.19990	.21530	.46186	.62666	.08636
- .20	1.16004	1.18396	1.18801	1.18871	.21718	.46443	.62893	.08685
- .19	1.15199	1.17367	1.17714	1.17770	.21909	.46698	.63115	.08736
- .18	1.14402	1.16351	1.16646	1.16694	.22100	.46953	.63335	.08786
- .17	1.13599	1.15345	1.15593	1.15627	.22294	.47205	.63550	.08837
- .16	1.12799	1.14347	1.14556	1.14584	.22488	.47450	.63762	.08888
- .15	1.12000	1.13365	1.13533	1.13555	.22683	.47703	.63968	.08940
- .14	1.11199	1.12388	1.12528	1.12542	.22879	.47949	.64171	.08991
- .13	1.10395	1.11425	1.11537	1.11545	.23076	.48191	.64370	.09043
- .12	1.09600	1.10477	1.10567	1.10569	.23276	.48433	.64566	.09096
- .11	1.08800	1.09541	1.09606	1.09610	.23475	.48672	.64757	.09148
- .10	1.08000	1.08612	1.08662	1.08666	.23675	.48906	.64944	.09201

f	$-3f[\mu_1^{(3)}]^2$	$-3f[\mu_1^{(4)}]^2$	$-3f[\mu_1^{(5)}]^2$	$-3f[\mu_1^{(6)}]^2$	$[\mu_2^{(4)}]^2$	$[\mu_2^{(5)}]^2$	$[\mu_2^{(6)}]^2$	$[\mu_3^{(6)}]^2$
-.10	1.08000	1.08612	1.08662	1.08666	.23675	.48906	.64944	.09201
-.09	1.07199	1.07697	1.07732	1.07736	.23877	.49140	.65128	.09254
-.08	1.06400	1.06794	1.06818	1.06820	.24079	.49370	.65307	.09308
-.07	1.05598	1.05901	1.05919	1.05919	.24281	.49598	.65485	.09362
-.06	1.04798	1.05022	1.05032	1.05032	.24485	.49824	.65657	.09416
-.05	1.03999	1.04154	1.04160	1.04160	.24689	.50046	.65826	.09471
-.04	1.03199	1.03299	1.03303	1.03303	.24893	.50264	.65991	.09525
-.03	1.02400	1.02455	1.02457	1.02457	.25098	.50480	.66152	.09580
-.02	1.01600	1.01625	1.01625	1.01625	.25303	.50694	.66310	.09636
-.01	1.00800	1.00806	1.00806	1.00806	.25509	.50904	.66465	.09692
.00	1.00000	1.00000	1.00000	1.00000	.25714	.51111	.66617	.09748
.01	.99200	.99206	.99206	.99206	.25920	.51316	.66764	.09804
.02	.98400	.98424	.98424	.98424	.26126	.51517	.66909	.09860
.03	.97601	.97656	.97654	.97654	.26331	.51715	.67050	.09917
.04	.96800	.96898	.96894	.96896	.26537	.51909	.67189	.09974
.05	.96001	.96154	.96148	.96150	.26742	.52102	.67324	.10031
.06	.95201	.95422	.95412	.95414	.26948	.52290	.67457	.10089
.07	.94401	.94702	.94687	.94688	.27152	.52477	.67586	.10147
.08	.93602	.93995	.93972	.93974	.27357	.52660	.67713	.10205
.09	.92800	.93298	.93265	.93269	.27561	.52840	.67838	.10263
.10	.92001	.92616	.92571	.92575	.27764	.53017	.67959	.10321
.11	.91202	.91943	.91888	.91891	.27967	.53192	.68077	.10380
.12	.90402	.91285	.91212	.91218	.28170	.53363	.68195	.10439
.13	.89599	.90638	.90545	.90552	.28371	.53531	.68307	.10498
.14	.88799	.89999	.89886	.89897	.28571	.53697	.68419	.10557
.15	.87999	89376	.89236	.89251	.28771	.53861	.68527	.10617
.16	.87200	.88763	.88595	.88616	.28970	.54021	.68635	.10677
.17	.86399	.88159	.87964	.87988	.29168	.54178	.68737	.10737
.18	.85601	.87570	.87342	.87368	.29364	.54333	.68840	.10796
.19	.84798	.86989	.86723	.86756	.29560	.54485	.68940	.10857
.20	.83999	.86425	.86115	.86156	.29754	54634	.69039	.10916
.21	.83200	.85868	.85513	.85562	.29947	.54781	.69134	.10977
.22	.82399	.85320	.84916	.84975	.30139	.54926	.69229	.11038
.23	.81599	.84784	.84327	.84399	.30329	.55067	.69321	.11099
.24	.80800	.84260	.83748	.83829	.30518	.55206	.69411	.11160
.25	.80001	.83744	.83174	.83267	.30706	.55343	.69499	.11221
.26	.79199	.83240	.82606	.82714	.30892	.55477	.69584	.11282
.27	.78400	.82743	.82046	.82165	.31077	.55610	.69669	.11343
.28	.77600	.82259	.81490	.81628	.31260	.55741	.69753	.11405
.29	.76801	.81783	.80939	.81095	.31442	.55868	.69834	.11466
.30	.76000	.81315	.80395	.80570	.31622	.55994	.69913	.11528
.31	.75200	.80861	.79856	.80052	.31801	.56117	.69992	.11589
.32	.74401	.80414	.79324	.79541	.31978	.56238	.70069	.11651
.33	.73601	.79974	.78794	.79037	.32152	.56358	.70142	.11713
.34	.72800	.79543	.78271	.78540	.32327	.56475	.70218	.11775
.35	.72002	.79125	.77754	.78050	.32499	.56590	.70288	.11836
.36	.71200	.78711	.77239	.77567	.32669	.56702	.70360	.11898

TABLE III

f	$-3f[\mu_1^{(3)}]^2$	$-3f[\mu_1^{(4)}]^2$	$-3f[\mu_1^{(5)}]^2$	$-3f[\mu_1^{(6)}]^2$	$[\mu_2^{(4)}]^2$	$[\mu_2^{(5)}]^2$	$[\mu_2^{(6)}]^2$	$[\mu_3^{(6)}]^2$
.36	.71200	.78711	.77239	.77567	.32669	.56702	.70360	.11898
.37	.70400	.78307	.76731	.77090	.32837	.56814	.70429	.11960
.38	.69601	.77911	.76227	.76619	.33005	.56924	.70498	.12022
.39	.68800	.77523	.75727	.76155	.33170	.57030	.70565	.12084
.40	.68001	.77143	.75231	.75695	.33333	.57137	.70631	.12145
.41	.67199	.76771	.74738	.75243	.33495	.57240	.70694	.12207
.42	.66400	.76407	.74251	.74797	.33655	.57343	.70758	.12269
.43	.65600	.76047	.73767	.74356	.33813	.57443	.70821	.12331
.44	.64799	.75697	.73289	.73922	.33970	.57541	.70881	.12393
.45	.64000	.75355	.72812	.73493	.34124	.57638	.70942	.12455
.46	.63199	.75016	.72340	.73072	.34278	.57734	.71001	.12516
.47	.62399	.74686	.71870	.72653	.34429	.57828	.71058	.12578
.48	.61601	.74365	.71406	.72243	.34578	.57921	.71115	.12639
.49	.60801	.74049	.70944	.71837	.34726	.58011	.71171	.12701
.50	.59999	.73737	.70484	.71436	.34873	.58101	.71225	.12762
.51	.59199	.73433	.70028	.71041	.35017	.58189	.71279	.12824
.52	.58400	.73135	.69576	.70652	.35160	.58276	.71333	.12885
.53	.57601	.72843	.69128	.70268	.35301	.58360	.71386	.12946
.54	.56800	.72556	.68679	.69888	.35441	.58444	.71436	.13006
.55	.56000	.72276	.68236	.69514	.35578	.58527	.71487	.13068
.56	.55200	.72000	.67795	.69146	.35714	.58608	.71536	.13128
.57	.54399	.71731	.67356	.68780	.35849	.58688	.71585	.13189
.58	.53600	.71465	.66921	.68421	.35982	.58766	.71633	.13249
.59	.52799	.71205	.66488	.68066	.36113	.58844	.71680	.13309
.60	.52000	.70950	.66056	.67718	.36243	.58921	.71726	.13369
.61	.51200	.70701	.65628	.67373	.36371	.58995	.71771	.13429
.62	.50400	.70456	.65202	.67032	.36497	.59068	.71817	.13489
.63	.49600	.70214	.64778	.66697	.36622	.59142	.71861	.13548
.64	.48800	.69980	.64357	.66365	.36745	.59213	.71904	.13608
.65	.48000	.69748	.63938	.66038	.36867	.59285	.71948	.13667
.66	.47200	.69521	.63521	.65715	.36988	.59355	.71988	.13726
.67	.46399	.69297	.63106	.65398	.37106	.59423	.72031	.13785
.68	.45600	.69081	.62693	.65085	.37223	.59490	.72072	.13844
.69	.44800	.68865	.62281	.64775	.37339	.59555	.72111	.13902
.70	.44001	.68654	.61872	.64470	.37454	.59622	.72151	.13960
.71	.43199	.68447	.61464	.64168	.37568	.59685	.72191	.14018
.72	.42400	.68244	.61059	.63870	.37680	.59750	.72228	.14075
.73	.41600	.68044	.60655	.63578	.37789	.59812	.72265	.14133
.74	.40800	.67850	.60253	.63290	.37899	.59874	.72303	.14190
.75	.40001	.67657	.59853	.63004	.38006	.59934	.72340	.14248
.76	.39200	.67468	.59455	.62722	.38113	.59994	.72378	.14304
.77	.38400	.67284	.59058	.62445	.38218	.60055	.72412	.14360
.78	.37600	.67102	.58663	.62172	.38322	.60112	.72447	.14417
.79	.36800	.66922	.58269	.61901	.38424	.60169	.72481	.14473
.80	.36000	.66746	.57877	.61633	.38526	.60225	.72515	.14528
.81	.35200	[illegible]	[illegible]	[illegible]	[illegible]	.60283	.72550	.14584

f	$-3f[\mu_1^{(3)}]^2$	$-3f[\mu_1^{(4)}]^2$	$-3f[\mu_1^{(5)}]^2$	$-3f[\mu_1^{(6)}]^2$	$[\mu_2^{(4)}]^2$	$[\mu_2^{(5)}]^2$	$[\mu_2^{(6)}]^2$	$[\mu_3^{(6)}]^2$
.81	.35200	.66574	.57487	.61372	.38625	.60283	.72550	.14584
.82	.34399	.66405	.57096	.61112	.38724	.60337	.72584	.14639
.83	.33599	.66237	.56708	.60856	.38822	.60392	.72616	.14694
.84	.32800	.66073	.56322	.60603	.38918	.60446	.72647	.14749
.85	.31999	.65912	.55937	.60354	.39014	.60497	.72679	.14803
.86	.31200	.65754	.55553	.60109	.39108	.60550	.72710	.14858
.87	.30400	.65599	.55171	.59866	.39200	.60602	.72742	.14911
.88	.29600	.65445	.54791	.59626	.39292	.60653	.72773	.14965
.89	.28800	.65293	.54410	.59390	.39382	.60703	.72802	.15019
.90	.28000	.65146	.54031	.59158	.39472	.60753	.72833	.15071
.91	.27200	.65001	.53654	.58927	.39560	.60801	.72862	.15124
.92	.26400	.64856	.53277	.58700	.39648	.60849	.72891	.15176
.93	.25600	.64716	.52902	.58477	.39734	.60898	.72918	.15229
.94	.24800	.64577	.52529	.58255	.39820	.60945	.72947	.15280
.95	.24000	.64441	.52154	.58037	.39904	.60990	.72974	.15332
.96	.23201	.64306	.51784	.57822	.39988	.61037	.73002	.15384
.97	.22400	.64173	.51412	.57610	.40069	.61080	.73029	.15434
.98	.21600	.64043	.51042	.57400	.40151	.61126	.73056	.15485
.99	.20800	.63915	.50673	.57194	.40231	.61170	.73082	.15535
1.00	.20000	.63789	.50305	.56990	.40311	.61213	.73108	.15585
1.1	.12000	.62631	.46676	.55089	.41057	.61619	.73352	.16070
1.2	.04000	.61632	.43127	.53419	.41723	.61979	.73563	.16523
1.3		.60762	.39643	.51944	.42320	.62301	.73761	.16948
1.4		.60001	.36214	.50639	.42858	.62590	.73932	.17345
1.5		.59329	.32832	.49477	.43342	.62852	.74089	.17714
1.6		.58731	.29488	.48442	.43783	.63088	.74230	.18058
1.7		.58197	.26178	.47513	.44184	.63303	.74360	.18379
1.8		.57719	.22897	.46679	.44552	.63500	.74477	.18678
1.9		.57287	.19642	.45925	.44887	.63680	.74586	.18957
2.0		.56894	.16409	.45240	.45197	.63846	.74686	.19219
2.1		.56537	.13194	.44618	.45482	.63998	.74778	.19462
2.2		.56211	.09998	.44051	.45746	.64141	.74862	.19691
2.3		.55912	.06816	.43532	.45991	.64272	.74942	.19904
2.4		.55635	.03648	.43055	.46220	.64396	.75016	.20105
2.5		.55380	.00492	.42615	.46432	.64510	.75086	.20294
3.0		.54354		.40855	.47308		.75372	.21087
4.0		.53054		.38671	.48468		.75753	.22166
5.0		.52266		.37379	.49199		.75997	.22860
6.0		.51739		.36528	.49700		.76164	.23340
7.0		.51362		.35927	.50064		.76286	.23693
8.0		.51078		.35481	.50343		.76381	.23961
9.0		.50858		.35135	.50561		.76456	.24173
10.0		.50682		.34862	.50737		.76515	.24343
∞		.49091		.32468	.52381		.77085	.25946

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